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# A POSSIBILISTIC OPTIMIZATION OVER AN INTEGER EFFICIENT SET WITHIN A FUZZY ENVIRONMENT

Assia Menni* and Djamal Chaabane

**Abstract.** Optimizing a linear function over the efficient set of a Multiple Objective Integer Linear Programming (MOILP) problem is known as a difficult problem to deal with, since a discrete efficient set is generally not convex and not explicitly known. Such problem becomes more and more difficult when parameters are defined with uncertainty. In this work, we deal with problems of this type for which parameters are imprecise and are assumed to be trapezoidal fuzzy numbers. The method is based on possibility and necessity measures introduced in the literature by D. Dubois and H. Prade.

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## 1. Introduction

Multiple Objective Optimization problems are often encountered in real world problems modeling. The parameters involved can be uncertain and make the situation complicated. As far as we know, there is no specific method that solves this kind of problems without switching to a deterministic mono-objective models. This doesn't mean eliminating uncertainty but, on the contrary, trying to preserve it all along the resolution process. By uncertainty we mean "randomness" and/or "fuzziness".

Fuzzy sets theory was born in 1965 [29], when L.A. Zadeh defined a fuzzy set as "a class" of objects with a continuum of grades membership and characterized it by a membership function, instead of classical characteristic function. As for fuzzy mathematical programming, Bellman and Zadeh [2], Tanaka *et al.* [25, 26] and Zimmermann [30] were the first to develop it. Zimmermann [31], suggest converting The Fuzzy MOLP problem into a single objective optimization problem (Max–Min problem) by using Bellman and Zadeh's fuzzy decision model (in [2]). In this approach, both of decision goals and the decision constraints should be satisfied, then the joint decision is to be optimized. Sakawa *et al.* [21], propose an interactive method for a Fuzzy MOLP problem with fuzzy goals of the DM and define the concept of $M$-Pareto optimal solution in terms of membership functions instead of objective functions. The DM has then to specify a reference membership value for each fuzzy objective function and a $M$-Pareto optimal solution is obtained by minimizing the difference between membership functions and their reference membership levels using $L_\infty$ norm. This approach is interactive in the sense that the reference levels can be changed from one iteration to another, as well as the membership

*Keywords.* Multiple objective programming, discrete optimization, Fuzzy parameters, possibility measure, necessity measure.

AMCD-RO-DGRSDT Laboratory, Faculty of Mathematics, Department of Operations Research, USTHB, Bab-Ezzouar, Bp 32 El Alia, BEZ, 16121 Algiers, Algeria.

*Corresponding author: assia.menni@gmail.com

functions of fuzzy goals. Subsequently, some extensions of these two approaches have been proposed in the literature. Both approaches are investigated on two phases: the fuzzy model is first converted to a deterministic one, then an optimization technique is applied to solve it. Inuiguchi and Sakawa [15] investigated an approach on a third phase in which they examine the efficiency of the obtained solution and they define two kinds of efficient solutions: possibly and necessarily efficient solutions.

In some situations, if the DM would prefer an efficient solution that optimizes a particular function of decision variables, we are therefore faced to an optimization problem over an efficient set of an MOLP problem (see *e.g.* [4, 5, 12, 14, 16, 19, 28]). With integer decision variables, the set of feasible solutions is no more convex. This frame of research includes few methods for which branch and bound techniques or/and Branch and cut algorithms are imposed [1, 7, 8, 17]. As for such a problem with stochastic parameters, an exact method has been developed in [6]. In this paper, we investigate the problem of optimizing a linear function over an efficient set of a Multiple Objective Integer Linear Programming (MOILP) problem within a fuzzy environment using possibility and necessity measures. This study can be motivated by some situations in real life problems. As an example, consider a production scheduling problem in which five different types of products are manufactured by an industrial firm: The fabrication process requires that each product pass through four workshops and the man-hours in each cannot be stated precisely but are described in a fuzzy way: in the $i$th workshop, the man-hour spent in manufacturing type $j$ is a fuzzy interval $\tilde{a}_{ij}$ ($i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3, 4, 5\}$). The overall man-hours in the $i$th workshop must not exceed a certain duration in a fuzzy time interval $\tilde{b}_i$.

The unit profits and costs of these five products are also described as fuzzy numbers $\{\tilde{c}_{11}, \tilde{c}_{12}, \tilde{c}_{13}, \tilde{c}_{14}, \tilde{c}_{15}\}$ and $\{\tilde{c}_{21}, \tilde{c}_{22}, \tilde{c}_{23}, \tilde{c}_{24}, \tilde{c}_{25}\}$ respectively, because they cannot be stated precisely in nature.

Let $x \in \mathbb{N}^5$ be a decision vector whose the $j$th component represents the number of the produced units of type $j$. The profit and cost resulting from a production plan $x$ are respectively given by $f_1(x) = \tilde{c}_1 x$ and $f_2(x) = \tilde{c}_2 x$.

The firm knows that manufacturing the five products produces toxic waste whose quantity can be estimated as fuzzy intervals $(\tilde{d}_1, \ldots, \tilde{d}_5)$ and wants to find a minimum-pollution production plan. However, the firm also seeks to maintain a high-level compromise between overall manufacturing cost and profit. In this case, a problem of type (2.3)(see Sect. 2) would be quite useful: Instead of minimizing $\tilde{d}x$ over the feasible set $X$, the firm would minimize $\tilde{d}x$ over the set of efficient solutions of a problem of type (2.4).

The next section contains some important results in fuzzy theory. In Section 3, we first state about a well known possibilistic technique that allows to transform fuzzy constraints of the fuzzy MOILP problem into deterministic ones using possibility and necessity measures, then we suggest a possibilistic way to overcome the fuzzy aspect of the main objective in the considered problem. In Section 4, we define fuzzy dominance and fuzzy efficiency with respect to possibility and necessity measures. An efficiency test in this sense is developed in the same section. In Section 5 we present a solving algorithm. An illustrative example is given in Section 6. Numerical results are given in Section 7 and the paper is concluded in Section 8.

## 2. Notations and main results

In ordinary sets theory, an element $x \in X$ is either in a given set $A \subset X$ or not. In fuzzy sets theory, the concept of membership of a fuzzy subset $\tilde{A}$ is generalized: $x$ belongs to $\tilde{A}$ with a degree of membership $\mu_{\tilde{A}}(x) \in [0, 1]$, where the extreme values $\mu_{\tilde{A}}(x) = 0$ and $\mu_{\tilde{A}}(x) = 1$ correspond to nonmembership and full membership respectively. Hence, the support of $\tilde{A}$ is defined as the ordinary subset for which the membership degrees of elements are positive while its kernel is defined as the ordinary subset for which the membership degree of elements is the largest one. An $\alpha$-cut of $\tilde{A}$, denoted by $\tilde{A}_\alpha$, is defined as the ordinary subset of $\tilde{A}$ for which the membership degrees of elements are greater than or equal to $\alpha$ ($\alpha \in [0, 1]$). The height of a fuzzy set $\tilde{A}$ is equal to the largest value of membership degrees of its elements. If the height of $\tilde{A}$ is 1 then $\tilde{A}$ is said to be normal. A fuzzy set $\tilde{A}$ for which $\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$ for all $x_1$ and $x_2$ in $\tilde{A}$ and $\lambda \in [0, 1]$, is said to be convex. A fuzzy number is a fuzzy subset of real numbers which is normal and convex (see *e.g.* [10, 20, 22]).

### 2.1. The problem

A classical problem of optimizing a linear function $\phi$ over the efficient set of a MOILP problem is generally formulated as follows:

$$(P_E) : \max\{\phi(x) = d^T x \mid x \in E(P)\} \tag{2.1}$$

where $E(P)$ denotes the set of efficient solutions of a MOILP problem $(P)$ defined in general by

$$(P)\begin{cases} \text{“max”}\ f_k(x) = c_k x\ ;\ \ k = 1, \ldots, p \\ x \in X = \{s \in \mathbb{Z}^n \mid As \leq b\,,\ s \geq 0\} \end{cases}. \tag{2.2}$$

$A$ is a $(m \times n)$ matrix, $b$ is a $m$-dimensionnal vector and $c_k$ is the $k$th row-vector of a $(p \times n)$ matrix $C$. In this paper we deal with the following optimization problem:

$$(\tilde{P}_E) : \max\{\tilde{\phi}(x) = \tilde{d}x \mid x \in E(\tilde{P})\} \tag{2.3}$$

where

$$(\tilde{P})\begin{cases} \text{“max”}\ \tilde{f}_k(x) = \tilde{c}_k x\ ;\ \ k = 1, \ldots, p \\ x \in \tilde{X} = \{s \in \mathbb{Z}^n \mid \tilde{A}s \leq \tilde{b}\,,\ s \geq 0\} \end{cases}. \tag{2.4}$$

Elements $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_{kj}$ and $\tilde{d}_j$ of $\tilde{A}$, $\tilde{b}$, $\tilde{C}$ and $\tilde{d}$ respectively are fuzzy numbers with membership functions $\mu_{\tilde{a}_{ij}}$, $\mu_{\tilde{b}_i}$, $\mu_{\tilde{c}_{kj}}$ and $\mu_{\tilde{d}_j}$ respectively ($i = 1, \ldots, m$, $j = 1, \ldots, n$ and $k = 1, \ldots, p$).

We consider here the situation in which all parameters of $(\tilde{P})$ and $(\tilde{P}_E)$ are trapezoidal fuzzy numbers. If $\tilde{N}$ is one of these coefficients, its membership function is then defined as [13,32]:

$$\mu_{\tilde{N}}(x) = \begin{cases} 0 & \text{if} \quad x \leq n^L - \sigma_N^L \text{ or } x \geq n^R + \sigma_N^R \\ 1 - \frac{n^L - x}{\sigma_N^L} & \text{if} \quad n^L - \sigma_N^L \leq x \leq n^L \\ 1 - \frac{x - n^R}{\sigma_N^R} & \text{if} \quad n^R \leq x \leq n^R + \sigma_N^R \\ 1 & \text{if} \quad n^L \leq x \leq n^R \end{cases} \tag{2.5}$$

such that

$$\mu_{\tilde{N}}(n^L - \sigma_N^L) = \begin{cases} 1 & \text{if} \quad \sigma_N^L = 0 \\ 0 & \text{if} \quad \sigma_N^L > 0 \end{cases} \quad \text{and} \quad \mu_{\tilde{N}}(n^R + \sigma_N^R) = \begin{cases} 1 & \text{if} \quad \sigma_N^R = 0 \\ 0 & \text{if} \quad \sigma_N^R > 0 \end{cases}$$

where $n^L$ and $n^R$ are respectively the lower and upper modal values of $\tilde{N}$ (see Fig. 1). $\sigma_N^L$ and $\sigma_N^R$ are real nonnegative values that represent Left and Right spreads of $\tilde{N}$ respectively. $\tilde{N}$ is symbolically denoted by

$$\tilde{N} = (n^L, n^R, \sigma_N^L, \sigma_N^R).$$

## 3. Possibility-based defuzzification of the problem

To compare two $LR$-flat fuzzy numbers $\tilde{M}$ and $\tilde{N}$, Dubois and Prade [10,11] give an index quantifying the possibility that $\tilde{M}$ is greater than or equal to $\tilde{N}$:

$$\Pi(\tilde{M} \geq \tilde{N}) = \sup_{m \geq n} \min\{\mu_{\tilde{M}}(m), \mu_{\tilde{N}}(n)\}. \tag{3.1}$$

As mentioned in [22] and [13], Figure 1 shows us that if the kernel of $\tilde{M}$ is located to the left of the kernel of $\tilde{N}$ then $\tilde{M}$ is less than or equal to $\tilde{N}$ with a possibility value equal to 1, but the possibility that $\tilde{M}$ is greater than or equal to $\tilde{N}$ is equal to the Height $h_0$ of the intersection of $\tilde{M}$ with $\tilde{N}$:

$$\Pi(\tilde{M} \geq \tilde{N}) = H(\tilde{M} \cap \tilde{N}) = h_0.$$

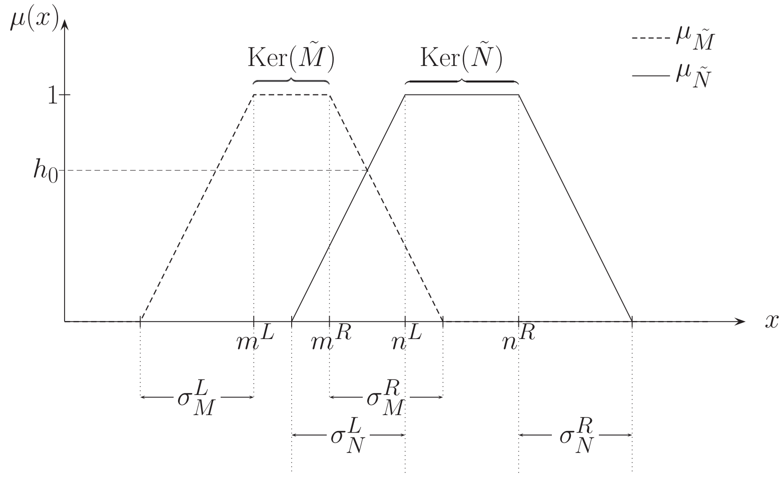


FIGURE 1. Inequality between two fuzzy numbers with respect to possibility measure.

Hence, if $ker(\tilde{M}) \cap ker(\tilde{N}) \neq \varnothing$ then $\Pi(\tilde{N} \geq \tilde{M}) = \Pi(\tilde{M} \geq \tilde{N}) = 1$.

Seen from this point of view, the possibility function measuring the possibility that $\tilde{M}$ is greater than or equal to another trapezoidal fuzzy number $\tilde{N}$, is defined as:

$$\Pi(\tilde{M} \geq \tilde{N}) = \begin{cases} 1 & \text{if} \quad n^L - m^R \leq 0 \\ 1 - \frac{n^L - m^R}{\sigma_M^R + \sigma_N^L} & \text{if} \quad 0 \leq n^L - m^R \leq \sigma_M^R + \sigma_N^L \\ 0 & \text{if} \quad n^L - m^R \geq \sigma_M^R + \sigma_N^L \end{cases}. \tag{3.2}$$

In the particular case where $\sigma_M^R = \sigma_N^L = 0$, we have $\Pi(\tilde{M} \geq \tilde{N}) = \begin{cases} 1 & \text{if} \quad n^L \leq m^R \\ 0 & \text{if} \quad n^L > m^R \end{cases}.$

Given a fixed level $\alpha$, we have

$$\Pi(\tilde{M} \geq \tilde{N}) \geq \alpha \Leftrightarrow 1 - \frac{n^L - m^R}{\sigma_M^R + \sigma_N^L} \geq \alpha \tag{3.3}$$

$$\Leftrightarrow n^L - (1-\alpha)\sigma_N^L \leq m^R + (1-\alpha)\sigma_M^R. \tag{3.4}$$

Now, as all fuzzy parameters of $(\tilde{P})$ are here assumed to be trapezoidal and by using the binary relation (3.4), the possibility that the $i$th constraint is satisfied is greater than or equal to $\alpha$ if and only if the following deterministic inequality is satisfied:

$$\sum_{j=1}^{n} \left( a_{ij}^L - (1-\alpha)\sigma_{a_{ij}}^L \right) x_j \leq b_i^R + (1-\alpha)\sigma_{b_i}^R \tag{3.5}$$

where $\tilde{a}_{ij} = (a_{ij}^L, a_{ij}^R, \sigma_{a_{ij}}^L, \sigma_{a_{ij}}^R)$ and $\tilde{b}_i = (b_i^L, b_i^R, \sigma_{b_i}^L, \sigma_{b_i}^R)$ $(i = 1, \ldots, m \; ; \; j = 1, \ldots, n)$.

Let $\chi_\alpha$ be the following set:

$$\chi_\alpha = \{x \in \mathbb{R}^n \mid \sum_{j=1}^{n} \left( a_{ij}^L - (1-\alpha)\sigma_{a_{ij}}^L \right) x_j \leq b_i^R + (1-\alpha)\sigma_{b_i}^R \;,\; \forall i = 1, \ldots, m\}. \tag{3.6}$$

A solution $x \in \chi_\alpha$ is admissible for $(\tilde{P})$ with a degree of possibility greater than or equal to $\alpha$.

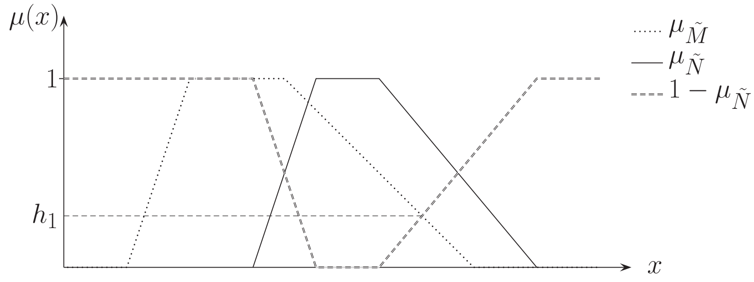


FIGURE 2. Strict inequality between two fuzzy numbers with respect to possibility measure.

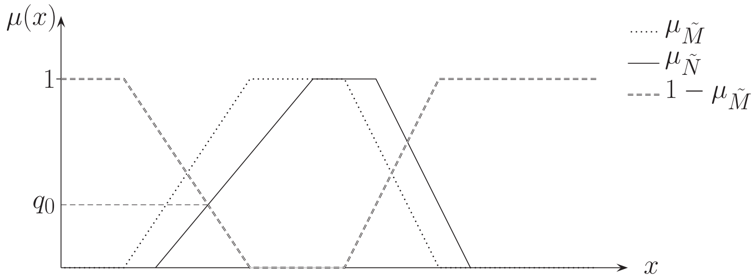


FIGURE 3. Inequality between two fuzzy numbers with respect to necessity measure.

In [11] authors give also an index quantifying the possibility that $\tilde{M}$ is strictly greater than $\tilde{N}$ (Fig. 2):

$$\Pi(\tilde{M} > \tilde{N}) = \sup_{m} \inf_{n \geq m} \min\left\{\mu_{\tilde{M}}(m), 1 - \mu_{\tilde{N}}(n)\right\} \tag{3.7}$$

$$\Pi(\tilde{M} > \tilde{N}) = h_1 = \begin{cases} 0 & \text{if} \quad m^R + \sigma_M^R \leq n^R \\ \frac{m^R + \sigma_M^R - n^R}{\sigma_M^R + \sigma_N^R} & \text{if} \quad n^R \leq m^R + \sigma_M^R \leq n^R + \sigma_M^R + \sigma_N^R \\ 1 & \text{if} \quad m^R \geq n^R + \sigma_N^R \end{cases}. \tag{3.8}$$

In the particular case where $\sigma_M^R = \sigma_N^R = 0$, we have: $\Pi(\tilde{M} > \tilde{N}) = \begin{cases} 0 & \text{if} \quad m^R \leq n^R \\ 1 & \text{if} \quad m^R > n^R \end{cases}.$

Always in the same context of comparing two fuzzy numbers, Dubois and Prade introduced also an index quantifying the necessity that a fuzzy number $\tilde{M}$ is greater than or equal to another fuzzy number $\tilde{N}$:

$$\mathcal{N}(\tilde{M} \geq \tilde{N}) = \inf_{m} \sup_{m \geq n} \max\left\{1 - \mu_{\tilde{M}}(m), \mu_{\tilde{N}}(n)\right\}. \tag{3.9}$$

A translation of equation (3.9) is given in Figure 3.

$$\mathcal{N}(\tilde{M} \geq \tilde{N}) = q_0 = \begin{cases} 0 & \text{if} \quad m^L \leq n^L - \sigma_N^L \\ \frac{m^L - n^L + \sigma_N^L}{\sigma_M^L + \sigma_N^L} & \text{if} \quad n^L - \sigma_N^L \leq m^L \leq n^L + \sigma_M^L \\ 1 & \text{if} \quad m^L - \sigma_M^L \geq n^L \end{cases}. \tag{3.10}$$

In the particular case where $\sigma_M^L = \sigma_N^L = 0$, we have: $\mathcal{N}(\tilde{M} \geq \tilde{N}) = \begin{cases} 0 & \text{if} \quad m^L < n^L \\ 1 & \text{if} \quad m^L \geq n^L \end{cases}.$

For a fixed level $\alpha$, we have:

$$\mathcal{N}(\tilde{M} \geq \tilde{N}) \geq \alpha \Leftrightarrow n^L - (1-\alpha)\sigma_N^L \leq m^L - \alpha\,\sigma_M^L \tag{3.11}$$

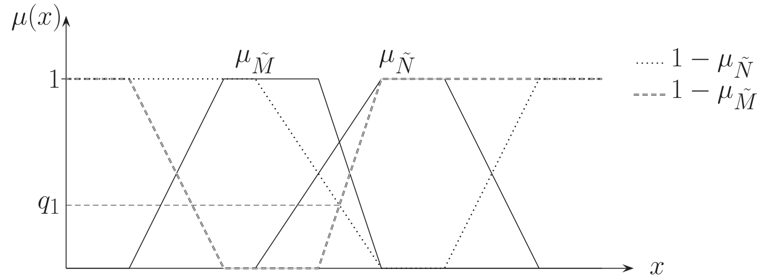


FIGURE 4. Strict inequality between two fuzzy numbers with respect to necessity measure.

and it can easily be proved that

$$\mathcal{N}(\tilde{M} \geq \tilde{N}) \geq \alpha \Rightarrow \Pi(\tilde{M} \geq \tilde{N}) \geq \alpha. \tag{3.12}$$

As for the $i$th fuzzy constraint of problem $(\tilde{P})$, the necessity that it is satisfied is greater than or equal to $\alpha$ if and only if the following crisp constraint is satisfied:

$$\sum_{j=1}^{n} \left(a_{ij}^{L} - (1-\alpha)\sigma_{a_{ij}}^{L}\right) x_j \leq b_i^L - \alpha\, \sigma_{b_i}^{L}\ ,\ i \in \{1, \ldots, m\}. \tag{3.13}$$

Now, let $\psi_\alpha$ be the following set:

$$\psi_\alpha = \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^{n} \left(a_{ij}^{L} - (1-\alpha)\sigma_{a_{ij}}^{L}\right) x_j \leq b_i^L - \alpha\, \sigma_{b_i}^{L}\ ,\ \forall i = 1, \ldots, m \right\}. \tag{3.14}$$

A solution $x \in \psi_\alpha$ is admissible for $(\tilde{P})$ with a degree of necessity greater than or equal to $\alpha$. From property 3.12, it's clear that $\psi_\alpha \subseteq \chi_\alpha$.

Also, the necessity that a fuzzy number $\tilde{M}$ is greater than another fuzzy number $\tilde{N}$ [11] is defined as (see Fig. 4):

$$\begin{aligned}\mathcal{N}(\tilde{M} > \tilde{N}) &= \inf_{n \geq m} \max\{1 - \mu_{\tilde{M}}(m), 1 - \mu_{\tilde{N}}(n)\} \\ &= 1 - \sup_{n \geq m} \min\{\mu_{\tilde{M}}(m), \mu_{\tilde{N}}(n)\} \\ &= 1 - \Pi(\tilde{N} \geq \tilde{M}).\end{aligned}$$

Now, as the coefficients $\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n$ are assumed to be trapezoidal fuzzy numbers, the main objective function $\tilde{\phi}$ is also trapezoidal (see for instance [9]):

$$\tilde{\phi}(x) = \left(\phi^L(x), \phi^R(x), \sigma^L_{\phi(x)}, \sigma^R_{\phi(x)}\right) \tag{3.15}$$

$$= \left(\sum_{j=1}^{n} d_j^L x_j, \sum_{j=1}^{n} d_j^R x_j, \sum_{j=1}^{n} \sigma_{d_j}^L x_j, \sum_{j=1}^{n} \sigma_{d_j}^R x_j\right). \tag{3.16}$$

In a possibilistic context, maximizing $\tilde{\phi}$ over $E(\tilde{P})$ consists on finding $x^\star \in E(\tilde{P})$ such that $\Pi\left(\tilde{\phi}(x^\star) \geq \tilde{\phi}(x)\right)$ is maximal for all $x \in E(\tilde{P})$. But it's well known that, given two $LR$-flat fuzzy numbers $\tilde{M}$ and $\tilde{N}$, we have $\max\{\Pi(\tilde{M} \geq \tilde{N}), \Pi(\tilde{N} \geq \tilde{M})\} = 1$. So, maximizing $\tilde{\phi}$ over $E(\tilde{P})$ consists on finding $x^\star \in E(\tilde{P})$ such that $\Pi\left(\tilde{\phi}(x^\star) \geq \tilde{\phi}(x)\right) = 1$ for all $x \in E(\tilde{P})$. In other words, if there doesn't exist another efficient solution $\bar{y} \in E(\tilde{P})$ for which $\Pi\left(\tilde{\phi}(x^\star) \geq \tilde{\phi}(\bar{y})\right) < 1$ then $x^\star$ is seen as a satisfactory solution for the decision-maker.

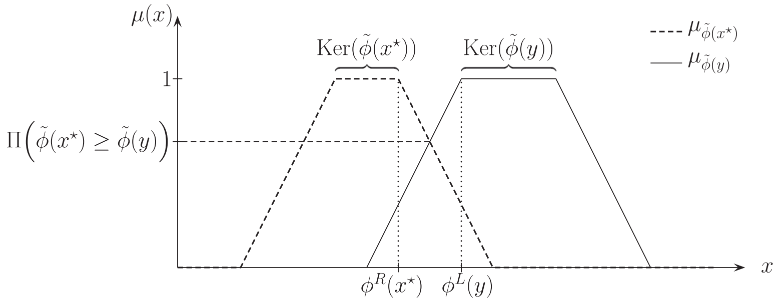


FIGURE 5. $\Pi\Big(\tilde{\phi}(x^\star) \geq \tilde{\phi}(y)\Big)$ is less than 1.

FIGURE 5. $\Pi\left(\tilde{\phi}(x^\star) \geq \tilde{\phi}(y)\right)$ is less than 1.

**Definition 3.1.** A solution $x^\star \in E(\tilde{P})$ is said to be satisfactory for problem $(\tilde{P}_E)$ if $\Pi\left(\tilde{\phi}(x^\star) \geq \tilde{\phi}(x)\right) = 1$ for all $x \in E(\tilde{P})$.

For the sake of simplicity, let's denote $\tilde{\phi}(x^\star)$ by $\tilde{G} = (g^L, g^R, \sigma_g^L, \sigma_g^R)$. Then we have $g^L = \phi^L(x^\star)$, $g^R = \phi^R(x^\star)$, $\sigma_g^L = \sigma^L_{\phi(x^\star)}$ and $\sigma_g^R = \sigma^R_{\phi(x^\star)}$

As shown above (see Fig. 1 with $\tilde{G}$ and $\tilde{\phi}(x)$ instead of $\tilde{M}$ and $\tilde{N}$ respectively), $\Pi\left(\tilde{G} \geq \tilde{\phi}(x)\right) = 1$ if the kernel of $\tilde{G}$ is located to the right of the kernel of $\tilde{\phi}(x)$ or if the intersection of the two kernels is not empty (*i.e.* if $g^R \geq \phi^L(x)$). As $\phi^R(x) \geq \phi^L(x)$, it can be affirmed that

$$g^R \geq \phi^R(x) \ , \ \forall x \in E(\tilde{P}) \Rightarrow \Pi\left(\tilde{G} \geq \tilde{\phi}(x)\right) = 1 \ , \ \forall x \in E(\tilde{P})$$

based upon, we suggest the following deterministic formulation of $(\tilde{P}_E)$:

$$(P_E)\begin{cases} \max \ \phi^R(x) = \displaystyle\sum_{j=1}^{n} d_j^R x_j \\ x \in E(\tilde{P}) \end{cases}.$$

**Lemma 3.2.** *An optimal solution $x^\star$ of $(P_E)$ is satisfactory for $(\tilde{P}_E)$. (see formula (2.3))*

*Proof.* Suppose $x^\star$ not being satisfactory for $(\tilde{P}_E)$. Then there exists $y \in E(\tilde{P})$ such that $\Pi\left(\tilde{\phi}(x^\star) \geq \tilde{\phi}(y)\right) < 1$. But the inequality $\tilde{\phi}(x^\star) \geq \tilde{\phi}(y)$ is not fully possible if and only if the upper bound of the kernel of $\tilde{\phi}(x^\star)$ is less than the lower bound of the kernel of $\tilde{\phi}(y)$ (see Fig. 5):

$$\Pi\left(\tilde{\phi}(x^\star) \geq \tilde{\phi}(y)\right) < 1 \Leftrightarrow \sum_{j=1}^{n} d_j^R x_j^\star < \sum_{j=1}^{n} d_j^L y_j.$$

Note here that $\sum_{j=1}^n d_j^R x_j^\star = \phi^R(x^\star)$ and $\sum_{j=1}^n d_j^L y_j = \phi^L(y)$ (formulas (3.15) and (3.16)). This implies that $\sum_{j=1}^n d_j^R x_j^\star < \sum_{j=1}^n d_j^R y_j$, since $\sum_{j=1}^n d_j^L y_j \leq \sum_{j=1}^n d_j^R y_j$ (by the very construction of flat–fuzzy numbers). The last strict inequality contradicts $x^\star$ being an optimal solution of $(P_E)$. $\square$

## 4. Efficiency seen from a Possibilistic viewpoint

**Definition 4.1** (Efficiency)**.** [3, 23, 27] An admissible solution $x^\star$ of $(P)$ is said to be efficient if and only if there doesn't exist another admissible solution $y$ of $(P)$ such that $f_k(y) \geq f_k(x^\star)$ for all $k \in \{1,\dots,p\}$ and $f_j(y) > f_j(x^\star)$ for at least one $j \in \{1,\dots,p\}$.

Let's denote by $\tilde{\Gamma}$ the fuzzy feasible set of $(\tilde{P})$ in objective space:

$$\tilde{\Gamma} = \{\tilde{z} = \tilde{C}x \,|\, x \in D_\alpha\}$$

where $D_\alpha$ is a defuzzified admissible set of $(\tilde{P})$. It is either $\chi_\alpha$ or $\psi_\alpha$ (see formulas (3.6) and (3.14)).

In a possibilistic context, we suggest the following definitions:

**Definition 4.2** ($\pi$-dominance)**.** A feasible fuzzy solution $\tilde{z} \in \tilde{\Gamma}$ of $(\tilde{P})$ is said to be $\pi$-dominated (*i.e.* possibly dominated) if and only if there exists another feasible fuzzy solution $\tilde{z}' \in \tilde{\Gamma}$ such that at least one of the following statements holds:

$(a)$ $\forall k \in \{1,\dots,p\}$, $\Pi\,(\tilde{z}_k > \tilde{z}'_k) = 0$ and $\exists l \in \{1,\dots,p\} \,|\, \Pi\,(\tilde{z}'_l > \tilde{z}_l) > 0$.
$(b)$ $\forall k \in \{1,\dots,p\}$, $\mathcal{N}\,(\tilde{z}'_k \geq \tilde{z}_k) = 1$ and $\exists l \in \{1,\dots,p\} \,|\, \mathcal{N}\,(\tilde{z}_l \geq \tilde{z}'_l) < 1$.

Let $x$ be an admissible solution of $(\tilde{P})$ such that $\tilde{z} = \tilde{C}x$. Formally, $\tilde{z}$ is $\pi$-dominated if and only if there exists $y \in D_\alpha$ such that at least one of the following holds:

(1). $\sum_{j=1}^{n} c^R_{kj} y_j \geq \sum_{j=1}^{n} (c^R_{kj} + \sigma^R_{c_{kj}}) x_j$ , $\forall k \in \{1,\dots,p\}$ and $\sum_{j=1}^{n} (c^R_{lj} + \sigma^R_{c_{lj}}) y_j > \sum_{j=1}^{n} c^R_{lj} x_j$ for at least one $l$.
(2). $\sum_{j=1}^{n} (c^L_{kj} - \sigma^L_{c_{kj}}) y_j \geq \sum_{j=1}^{n} c^L_{kj} x_j$ , $\forall k \in \{1,\dots,p\}$ and $\sum_{j=1}^{n} c^L_{lj} y_j > \sum_{j=1}^{n} (c^L_{lj} - \sigma^L_{c_{lj}}) x_j$ for at least one $l$.

**Definition 4.3** ($\mathcal{N}$-dominance)**.** A feasible fuzzy solution $\tilde{z} \in \tilde{\Gamma}$ of $(\tilde{P})$ is said to be $\mathcal{N}$-dominated (*i.e.* necessarily dominated) if and only if there exists $\tilde{w} \in \tilde{\Gamma}$ such that (a) holds and there exists $\tilde{y} \in \tilde{\Gamma}$ such that (b) holds (when replacing $\tilde{z}'$ by $\tilde{w}$ and $\tilde{y}$ in (a) and (b) respectively).

**Definition 4.4.** A feasible fuzzy solution $\tilde{z} \in \tilde{\Gamma}$ of $(\tilde{P})$ is said to be $\mathcal{N}$-dominated by another feasible fuzzy solution $\tilde{z}' \in \tilde{\Gamma}$ if and only if (a) and (b) hold simultaneously.

**Definition 4.5** ($\pi$-efficiency)**.** An admissible solution $x^\star \in D_\alpha$ is said to be $\pi$-efficient (*i.e.* possibly efficient) for $(\tilde{P})$ if and only if $\tilde{C}x^\star$ is not $\mathcal{N}$-dominated.

**Definition 4.6** ($\mathcal{N}$-efficiency)**.** An admissible solution $x^\star \in D_\alpha$ is said to be $\mathcal{N}$-efficient (*i.e.* necessarily efficient) for $(\tilde{P})$ if and only if $\tilde{C}x^\star$ is not $\pi$-dominated.

Let's denote by $E_\pi(\tilde{P})$ and $E_\mathcal{N}(\tilde{P})$ the sets of $\pi$-efficient solutions and $\mathcal{N}$-efficient solutions of $(\tilde{P})$ respectively and let's denote by $\Delta_\pi(\tilde{P})$ and $\Delta_\mathcal{N}(\tilde{P})$ the sets of $\pi$-dominated solutions and $\mathcal{N}$-dominated solutions of $(\tilde{P})$ respectively. The following property is obvious.

**Property 4.7.** We have $E_\mathcal{N}(\tilde{P}) \subseteq E_\pi(\tilde{P})$ and $\Delta_\mathcal{N}(\tilde{P}) \subseteq \Delta_\pi(\tilde{P})$.

Now, let $\hat{x}$ be a given admissible solution of $(\tilde{P})$. According to Definitions 4.2 and 4.3, the following two Linear Programs are formulated in order to examine $\pi$-efficiency and/or $\mathcal{N}$-efficiency of $\hat{x}$:

$$(T^L_{\{\hat{x}\}}) \begin{cases} \max\ \Theta^L = \sum_{k=1}^{p} \theta^L_k \\ \sum_{j=1}^{n} c^L_{kj} x_j - \theta^L_k = \sum_{j=1}^{n} (c^L_{kj} - \sigma^L_{c_{kj}}) \hat{x}_j \text{ , } \forall k = \overline{1,p} \\ \sum_{j=1}^{n} (c^L_{kj} - \sigma^L_{c_{kj}}) x_j \geq \sum_{j=1}^{n} c^L_{kj} \hat{x}_j \text{ , } \forall k = \overline{1,p} \\ x \in D_\alpha,\ \theta^L_k \geq 0,\ \forall k \in \{1,\dots,p\} \end{cases} ;\ (T^R_{\{\hat{x}\}}) \begin{cases} \max\ \Theta^R = \sum_{k=1}^{p} \theta^R_k \\ \sum_{j=1}^{n} (c^R_{kj} + \sigma^R_{c_{kj}}) x_j - \theta^R_k = \sum_{j=1}^{n} c^R_{kj} \hat{x}_j \text{ , } \forall k = \overline{1,p} \\ \sum_{j=1}^{n} c^R_{kj} x_j \geq \sum_{j=1}^{n} (c^R_{kj} + \sigma^R_{c_{kj}}) \hat{x}_j \text{ , } \forall k = \overline{1,p} \\ x \in D_\alpha,\ \theta^R_k \geq 0,\ \forall k \in \{1,\dots,p\} \end{cases}$$

where $\tilde{c}_{kj} = (c^L_{kj}, c^R_{kj}, \sigma^L_{c_{kj}}, \sigma^R_{c_{kj}})$ $(k = 1,\dots,p\,;\, j = 1,\dots,n)$ are the fuzzy objective coefficients.

By solving $(T^L_{\{\hat{x}\}})$, we look for an admissible solution $y$ that satisfies condition (2) of Definition 4.2. Such a solution doesn't exist if and only if the large inequality doesn't hold for at least one $k \in \{1,2,\dots,p\}$ (*i.e.* $(T^L_{\{\hat{x}\}})$ *is unfeasible*) or the strict inequality doesn't hold for all $l \in \{1,2,\dots,p\}$ (*i.e.* $(T^L_{\{\hat{x}\}})$ *is unfeasible or its optimal objective value is zero*). Similarly, by solving $(T^R_{\{\hat{x}\}})$, we look for an admissible solution that satisfies condition (1).

From Definitions 4.5 and 4.6, we obtain the following propositions:

**Proposition 4.8.** *An admissible solution $\hat{x} \in D_\alpha$ is $\pi$-efficient for $(\tilde{P})$ if and only if at least one of $(T^L_{\{\hat{x}\}})$ or $(T^R_{\{\hat{x}\}})$ is unfeasible or its optimal objective value is zero.*

*Proof.* Suppose that both $(T^L_{\{\hat{x}\}})$ and $(T^R_{\{\hat{x}\}})$ are feasible and let $\hat{\Theta}^L$ and $\hat{\Theta}^R$ be their optimal objective values respectively. It follows that:

$\hat{\Theta}^L > 0$ and $\hat{\Theta}^R > 0 \Leftrightarrow$ there exists $y \in D_\alpha$ such that

$$\sum_{j=1}^{n} c^R_{kj} y_j \geq \sum_{j=1}^{n} (c^R_{kj} + \sigma^R_{c_{kj}})\hat{x}_j\,,\ \forall k \in \{1,\dots,p\}$$

$$\sum_{j=1}^{n} (c^R_{lj} + \sigma^R_{c_{lj}}) y_j > \sum_{j=1}^{n} c^R_{lj}\hat{x}_j \text{ for at least one } l$$

and there exists $y' \in D_\alpha$ such that

$$\sum_{j=1}^{n} (c^L_{kj} - \sigma^L_{c_{kj}}) y'_j \geq \sum_{j=1}^{n} c^L_{kj}\hat{x}_j\,,\ \forall k \in \{1,\dots,p\}$$

$$\sum_{j=1}^{n} c^L_{qj} y'_j > \sum_{j=1}^{n} (c^L_{qj} - \sigma^L_{c_{qj}}) x_j \text{ for at least one } q$$

$$\Leftrightarrow \tilde{C}\hat{x} \text{ is } \mathcal{N}\text{-dominated}$$

$$\Leftrightarrow \hat{x} \text{ is not } \pi\text{-efficient.}$$

Hence, $\hat{x}$ is $\pi$-efficient if and only if $\hat{\Theta}^L = 0$ or $\hat{\Theta}^R = 0$ or $(T^L_{\{\hat{x}\}})$ is unfeasible or $(T^R_{\{\hat{x}\}})$ is unfeasible. □

**Proposition 4.9.** *A solution $\hat{x} \in D_\alpha$ is $\mathcal{N}$-efficient for $(\tilde{P})$ if and only if each of the two Linear Programs $(T^L_{\{\hat{x}\}})$ and $(T^R_{\{\hat{x}\}})$ is unfeasible or its optimal objective value is zero.*

*Proof.*

$\hat{x}$ is $\mathcal{N}$-efficient $\Leftrightarrow$ $\hat{x}$ is not $\pi$-dominated

$\Leftrightarrow \nexists y \in D_\alpha$ such that

$$\sum_{j=1}^{n} (c^L_{kj} - \sigma^L_{c_{kj}}) y_j \geq \sum_{j=1}^{n} c^L_{kj}\hat{x}_j\,,\ \forall k = 1,\dots,p \text{ and } \exists l \in \{1,\dots,p\} \mid \sum_{j=1}^{n} c^L_{lj} y_j > \sum_{j=1}^{n} (c^L_{lj} - \sigma^L_{c_{lj}})\hat{x}_j$$

and $\nexists y' \in D_\alpha$ such that

$$\sum_{j=1}^{n} c^R_{kj} y'_j \geq \sum_{j=1}^{n} (c^R_{kj} + \sigma^R_{c_{kj}})\hat{x}_j\,,\ \forall k = 1,\dots,p \text{ and } \exists l \in \{1,\dots,p\} \mid \sum_{j=1}^{n} (c^R_{lj} + \sigma^R_{c_{lj}}) y'_j > \sum_{j=1}^{n} c^R_{lj}\hat{x}_j$$

$$\Leftrightarrow \{(T^L_{\{\hat{x}\}}) \text{ is unfeasible or } \hat{\Theta}^L = 0\} \text{ and } \{(T^R_{\{\hat{x}\}}) \text{ is unfeasible or } \hat{\Theta}^R = 0\}.$$

□

We can summarize the decision rule for the nature of $\hat{x}$ in Table 1.

**Proposition 4.10.** *If $(T^L_{\{\hat{x}\}})$ has an optimal solution $(\hat{s}\,,\hat{\Theta}^L)$, such that $\hat{\Theta}^L > 0$, then $\hat{s}$ is $\pi$-efficient.*

TABLE 1. Decision rule for the nature of $\hat{x}$.

| | $(T^L_{\{\hat{x}\}})$ is unfeasible | $(T^L_{\{\hat{x}\}})$ has a null optimal objective value | $(T^L_{\{\hat{x}\}})$ has a positive optimal objective value |
|---|---|---|---|
| $(T^R_{\{\hat{x}\}})$ is unfeasible | $\hat{x}$ is $\mathcal{N}$-efficient | $\hat{x}$ is $\mathcal{N}$-efficient | $\hat{x}$ is $\pi$-efficient |
| $(T^R_{\{\hat{x}\}})$ has a null optimal objective value | $\hat{x}$ is $\mathcal{N}$-efficient | $\hat{x}$ is $\mathcal{N}$-efficient | $\hat{x}$ is $\pi$-efficient |
| $(T^R_{\{\hat{x}\}})$ has a positive optimal objective value | $\hat{x}$ is $\pi$-efficient | $\hat{x}$ is $\pi$-efficient | $\hat{x}$ is neither $\mathcal{N}$-efficient nor $\pi$-efficient |

*Proof.* Suppose that $\hat{s}$ is not $\pi$-efficient. Then $\tilde{C}\hat{s}$ is $\mathcal{N}$-dominated. Therefore, there exists $y \in D_\alpha$ and there exists $y' \in D_\alpha$ such that:

(1) $\sum_{j=1}^n (c^L_{kj} - \sigma^L_{c_{kj}}) y_j \geq \sum_{j=1}^n c^L_{kj}\hat{s}_j$ , $\forall k = 1, \ldots, p$.
(2) $\sum_{j=1}^n c^R_{kj} y'_j \geq \sum_{j=1}^n (c^R_{kj} + \sigma^R_{c_{kj}})\hat{s}_j$ , $\forall k = 1, \ldots, p$.
(3) $\sum_{j=1}^n c^L_{lj} y_j > \sum_{j=1}^n (c^L_{lj} - \sigma^L_{c_{lj}})\hat{s}_j$ for at least one index $l \in \{1, \ldots, p\}$.
(4) $\sum_{j=1}^n (c^R_{lj} + \sigma^R_{c_{lj}}) y'_j > \sum_{j=1}^n c^R_{lj}\hat{s}_j$ for at least one index $l \in \{1, \ldots, p\}$.

From inequality (1) we deduce that $\sum_{j=1}^n (c^L_{kj} - \sigma^L_{c_{kj}}) y_j \geq \sum_{j=1}^n c^L_{kj}\hat{x}_j$ $\forall k = 1, \ldots, p$, since we have

$$\sum_{j=1}^n c^L_{kj}\hat{s}_j \geq \sum_{j=1}^n \left(c^L_{kj} - \sigma^L_{c_{kj}}\right)\hat{s}_j \geq \sum_{j=1}^n c^L_{kj}\hat{x}_j \ , \ \forall k = 1, \ldots, p.$$

Inequality (3) (joined to inequality (1)) indicates that $\sum_{j=1}^n c^L_{kj} y_j \geq \sum_{j=1}^n c^L_{kj}\hat{s}_j$ for all $k \in \{1, \ldots, p\}$, with at least one strict inequality.

This implies that $\sum_{j=1}^n c^L_{kj} y_j - \sum_{j=1}^n (c^L_{kj} - \sigma^L_{c_{kj}})\hat{x}_j \geq \sum_{j=1}^n c^L_{kj}\hat{s}_j - \sum_{j=1}^n (c^L_{kj} - \sigma^L_{c_{kj}})\hat{x}_j = \hat{\theta}^L_k$ for all $k \in \{1, \ldots, p\}$, with at least one strict inequality.

Now, by summing over $k$ we get $\sum_{k=1}^p \sum_{j=1}^n \left(c^L_{kj} y_j - (c^L_{kj} - \sigma^R_{c_{kj}})\hat{x}_j\right) > \sum_{k=1}^p \hat{\theta}^L_k = \hat{\Theta}^L$.

The last strict inequality contradicts $(\hat{s}\,,\,\hat{\Theta}^L)$ being optimal for $(T^L_{\{\hat{x}\}})$. Thus, $\hat{s}$ is $\pi$-efficient. □

**Proposition 4.11.** *If $(T^R_{\{\hat{x}\}})$ has an optimal solution $(\hat{w}\,,\,\hat{\Theta}^R)$, such that $\hat{\Theta}^R > 0$, then $\hat{w}$ is $\pi$-efficient.*

*Proof.* By using inequalities (2) and (4) the proof is similar to that of Proposition 4.10. □

**Proposition 4.12.** *If the resolution of $(T^L_{\{\hat{x}\}})$ and $(T^R_{\{\hat{x}\}})$ provides a same optimal solution $\hat{S}$ with positive optimal objective values, then $\hat{S}$ is $\mathcal{N}$-efficient.*

*Proof.* Let's consider again the inequalities (1)–(4) that are involved in the proof of Proposition 4.10. We have shown that (1) and (3) imply that $\hat{s}$ is not optimal for $(T^L_{\{\hat{x}\}})$ and in the same manner, we can show that (2) and (4) imply that $\hat{w}$ is not optimal for $(T^R_{\{\hat{x}\}})$. By replacing $\hat{s}$ and $\hat{w}$ by $\hat{S}$ which is assumed to be optimal for both of $(T^L_{\{\hat{x}\}})$ and $(T^R_{\{\hat{x}\}})$, we have:

$$\begin{aligned} \hat{S} \text{ is not } \mathcal{N}\text{-efficient} &\Leftrightarrow \hat{S} \text{ is } \pi\text{-dominated} \\ &\Leftrightarrow \text{“}\exists y \in D_\alpha \text{ such that (1) and (3) hold” OR “}\exists y \in D_\alpha \text{ such that (2) and (4) hold”} \\ &\Rightarrow \hat{S} \text{ is not optimal for } (T^L_{\{\hat{x}\}}) \text{ OR } \hat{S} \text{ is not optimal for } (T^R_{\{\hat{x}\}}) \end{aligned}$$

which leads to a contradiction. □

## 5. An algorithm for optimizing $\tilde{\phi}$ over $E(\tilde{P})$

In the algorithm of finding an optimal solution to problem $(P_E)$ which is proposed in [7], the admissible set of problem $(P)$ is reduced iteratively by successive elimination of subsets of dominated solutions, until the test of efficiency of the current solution is positive. Sylva and Crema's cuts (see [24]) which are used in this elimination process are extended here to be used in a possibilistic framework to solve problem $(\tilde{P}_E)$.

### 5.1. Detailed description of the algorithm

In this section we give a detailed description of the proposed algorithm. If we are interested in optimizing $\tilde{\phi}$ over the set of $\pi$-efficient solutions of $(\tilde{P})$ (*i.e.* $E(\tilde{P}) = E_\pi(\tilde{P})$), then at each iteration of the searching process we use cuts which only keep a subset of admissible solutions not $\mathcal{N}$-dominating with the current one. If instead we are interested in finding a satisfactory solution of $\tilde{\phi}$ over the set of $\mathcal{N}$-efficient solutions of $(\tilde{P})$ (*i.e.* $E(\tilde{P}) = E_\mathcal{N}(\tilde{P})$), then at each iteration, we use cuts which only keep a subset of admissible solutions which do not $\pi$-dominate with the current one.

*5.1.1. Case where $E(\tilde{P}) = E_\pi(\tilde{P})$*

At a given iteration $t$ where a $\pi$-efficient solution $\hat{s}^t$ is detected, we reduce $D_\alpha$ as follows:

$$\mathcal{D}_t = \left\{ x \in \mathcal{D}_{t-1} \;\middle|\; \begin{array}{l} \Pi\left(\tilde{f}_k(x) > \lambda_k \tilde{f}_k(\hat{s}^t) + (1-\lambda_k)\tilde{L}_k^0\right) > 0 \;,\; \forall k = 1,\dots,p \\ \mathcal{N}\left(\gamma_k \tilde{f}_k(\hat{s}^t) + (1-\gamma_k)\tilde{L}_k^1 \geq \tilde{f}_k(x)\right) < 1 \;,\; \forall k = 1,\dots,p \\ \displaystyle\sum_{k=1}^{p} \lambda_k + \gamma_k \geq 1 \end{array} \right\}$$

where, $\mathcal{D}_0 = D_\alpha$ and $\lambda_k, \gamma_k \in \{0,1\}$ $\forall k = 1,\dots,p$

$\tilde{L}_k^0$ is a trapezoidal fuzzy number for which ${L_k^0}^R + \sigma^R_{L_k^0} = \mathfrak{L}_k^0 = \min_{x\in D_\alpha}\{\sum_{j=1}^n (c_{kj}^R + \sigma^R_{L_k})x_j\}$ ; $k = 1,\dots,p.$

$\tilde{L}_k^1$ is a trapezoidal fuzzy number for which ${L_k^1}^L = \mathfrak{L}_k^1 = \min_{x\in D_\alpha}\{\sum_{j=1}^n c_{kj}^L x_j\}$ ; $k = 1,\dots,p.$

Thus, $\mathcal{D}_t$ is defined as:

$$\mathcal{D}_t = \left\{ x \in \mathcal{D}_{t-1} \;\middle|\; \begin{array}{l} \displaystyle\sum_{j=1}^{n}(c_{kj}^R + \sigma^R_{c_{kj}})x_j \geq \lambda_k(\sum_{j=1}^{n} c_{kj}^R \hat{s}_j^t + \varepsilon) + (1-\lambda_k)\mathfrak{L}_k^0 \;\forall k = 1,\dots,p \\ \displaystyle\sum_{j=1}^{n} c_{kj}^L x_j \geq \gamma_k\left(\sum_{j=1}^{n}(c_{kj}^L - \sigma^L_{c_{kj}})\hat{s}_j^t + \varepsilon\right) + (1-\gamma_k)\mathfrak{L}_k^1 \;,\; \forall k = 1,\dots,p, \\ \displaystyle\sum_{k=1}^{p} \lambda_k + \gamma_k \geq 1,\; \lambda_k,\, \gamma_k \in \{0,1\} \end{array} \right\}$$

where $\varepsilon$ is a positive real number chosen to be small enough.

If $\lambda_k = 1$, the constraint $\sum_{j=1}^n (c_{kj}^R + \sigma^R_{c_{kj}})x_j > \sum_{j=1}^n c_{kj}^R \hat{s}_j^t$ implies that $\tilde{C}\hat{s}^t$ doesn't $\mathcal{N}$-dominate $\tilde{C}x$. So $\tilde{C}x$ is either $\mathcal{N}$-dominated by another solutions or $x$ is $\pi$-efficient.

If $\lambda_k = 0$, the constraint $\sum_{j=1}^n (c_{kj}^R + \sigma^R_{c_{kj}})x_j \geq \mathfrak{L}_k^0$ is always satisfied.

If $\gamma_k = 1$, the constraint $\sum_{j=1}^n c_{kj}^L x_j > \sum_{j=1}^n (c_{kj}^L - \sigma^L_{c_{kj}})\hat{s}_j^t$ implies that $\tilde{C}\hat{s}^t$ doesn't $\mathcal{N}$-dominate $\tilde{C}x$. So $\tilde{C}x$ is either $\mathcal{N}$-dominated by another solutions or $x$ is $\pi$-efficient.

If $\gamma_k = 0$, the constraint $\sum_{j=1}^n c_{kj}^L x_j \geq \mathfrak{L}_k^1$ is always satisfied.

The constraint $\sum_{k=1}^p \lambda_k + \gamma_k \geq 1$ means that at least one index $l \in \{1,\dots,p\}$ verifies $\lambda_l = 1$ or $\gamma_l = 1$. *i.e.* at least one necessarily dominance constraint is violated.

*5.1.2. Case where $E(\tilde{P}) = E_\mathcal{N}(\tilde{P})$*

At a given iteration $t$ where a $\pi$-efficient solution $\hat{s}^t$ is detected, we reduce $D_\alpha$ as follows:

$$\mathcal{D}_t = \left\{ x \in \mathcal{D}_{t-1} \;\middle|\; \begin{array}{l} \Pi\left(\tilde{f}_k(x) > \lambda_k \tilde{f}_k(\hat{s}^t) + (1-\lambda_k)\tilde{L}_k^0\right) > 0 \;,\; \forall k = 1,\dots,p \\ \mathcal{N}\left(\lambda_k \tilde{f}_k(\hat{s}^t) + (1-\lambda_k)\tilde{L}_k^1 \geq \tilde{f}_k(x)\right) < 1 \;,\; \forall k = 1,\dots,p \\ \displaystyle\sum_{k=1}^{p} \lambda_k \geq 1 \end{array} \right\}.$$

Hence, the admissible domain is iteratively defined by:

$$\mathcal{D}_t = \left\{ x \in \mathcal{D}_{t-1} \;\middle|\; \begin{array}{l} \sum_{j=1}^{n} (c_{kj}^R + \sigma_{c_{kj}}^R) x_j \geq \lambda_k (\sum_{j=1}^{n} c_{kj}^R \hat{s}_j^t + \varepsilon) + (1-\lambda_k)\mathfrak{L}_k^0 \; \forall k = 1,\dots,p \\ \sum_{j=1}^{n} c_{kj}^L x_j \geq \lambda_k \left( \sum_{j=1}^{n} (c_{kj}^L - \sigma_{c_{kj}}^L)\hat{s}_j^t + \varepsilon \right) + (1-\lambda_k)\mathfrak{L}_k^1 \;, \forall k = 1,\dots,p, \\ \sum_{k=1}^{p} \lambda_k \geq 1, \; \lambda_k \in \{0,1\} \end{array} \right\}$$

where $\varepsilon$ is a positive real number chosen close to zero.

If $\lambda_k = 1$, the constraints $\sum_{j=1}^{n} (c_{kj}^R + \sigma_{c_{kj}}^R) x_j > \sum_{j=1}^{n} c_{kj}^R \hat{s}_j^t$ and $\sum_{j=1}^{n} c_{kj}^L x_j > \sum_{j=1}^{n} (c_{kj}^L - \sigma_{c_{kj}}^L)\hat{s}_j^t$ imply that $\tilde{C}\hat{s}^t$ doesn't $\pi$-dominate $\tilde{C}x$. So $\tilde{C}x$ is either $\pi$-dominated by another solution or $x$ is $\mathcal{N}$-efficient.

If $\lambda_k = 0$, the constraints $\sum_{j=1}^{n} (c_{kj}^R + \sigma_{c_{kj}}^R) x_j \geq \mathfrak{L}_k^0$ and $\sum_{j=1}^{n} c_{kj}^L x_j \geq \mathfrak{L}_k^1$ are always satisfied.

## 5.2. Technical description of the algorithm

A technical presentation of the proposed method for optimizing over a possibilistic efficient set is given in the algorithm bellow where, at an iteration $t$, the programs to be solved are the following:

- $(P_t)$ : $\max\{\sum_{j=1}^{n} d_j^R x_j \mid x \in D_t\}$.
- $(T_t)$ is the efficiency test of an optimal solution $x^t$ of $(P_t)$.
- $(Q_t)$ : $\max\{\sum_{j=1}^{n} d_j^R x_j \mid \tilde{f}_k(x) = \tilde{f}_k(s^t) \forall k \in \{1,\dots,p\}$ and $x \in D_\alpha\}$ where $s^t$ is an optimal solution of $(T_t)$ if the solution tested is not efficient.

**Algorithm 1:** Optimizing $\tilde{\phi}$ over $E(\tilde{P})$.

**Input parameters**: ;
$\downarrow\tilde{A}_{(m\times n)}$: Matrix coefficients involved in the fuzzy constraints of $(\tilde{P})$.;
$\downarrow\tilde{b}_{(m\times 1)}$: Right Hand Side fuzzy vector.;
$\downarrow\tilde{C}_{(p\times n)}$: Matrix of fuzzy objective function's coefficients.;
$\downarrow\tilde{d}_{(1\times n)}$: Main Objective fuzzy Vector.;
**Output**: ;
$\uparrow x^{\text{opt}}$: Optimal Solution of the Main Problem $(\tilde{P}_E)$.;
$\uparrow\tilde{\phi}^{\text{opt}}$: Optimal Main Objective fuzzy Value.;
**Initialization**: research $\leftarrow$ true , $t \leftarrow 0$ , $\mathcal{D}_t \leftarrow D_\alpha$ , $\phi^{\text{opt}} \leftarrow -\infty$
**Searching for an Optimal Solution of** $(\tilde{P}_E)$: ;
Solve the Relaxed Problem $(P_t)$: $[\uparrow x^t , \uparrow\phi^t] = P_t(\downarrow d^R , \mathcal{D}_t)$;
**if** $(P_t)$ *is not feasible* **then**
  $(P_E)$ is not feasible
**else**
  **while** *research=true* **do**
    $T_t(\downarrow x^t)$: efficiency test;
    **if** $T_t(\downarrow x^t)$ *is positive* **then**
      $x^{\text{opt}} \leftarrow x^t$ , $\phi^{\text{opt}} \leftarrow \phi^t$ , research $\leftarrow$ false
    **else**
      $[\uparrow s_{eq}^t , \uparrow\phi_{eq}^t] = Q_t(\downarrow s^t , \downarrow d^R , \mathcal{D}_t)$;
      $x^{\text{opt}} \leftarrow s_{eq}^t$ , $\phi^{\text{opt}} \leftarrow \phi_{eq}^t$ , $t \leftarrow t+1$;
      $[\uparrow x^t , \uparrow\phi^t] = P_t(\downarrow d^R , \mathcal{D}_t)$;
      **if** $(P_t)$*is unfeasible OR* $\phi^t \leq \phi^{\text{opt}}$ **then**
        research $\leftarrow$ false
      **end**
    **end**
  **end**
**end**

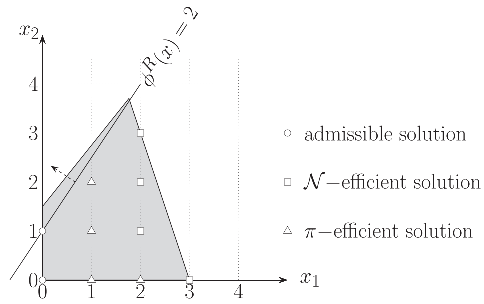


FIGURE 6. The search domain is $D_0 = D$ in gray color.

## 6. Illustrative example

Consider the following fuzzy bi-objective linear programming problem:

$$(\tilde{P}) \begin{cases} \text{“max”}\, \tilde{f}_1(x) = (2, 4, 0, \tfrac{1}{4})\, x_1 + (-3, -1, \tfrac{1}{4}, 0)\, x_2 \\ \text{“max”}\, \tilde{f}_2(x) = (-1, 1, 0, \tfrac{1}{4})\, x_1 + (3, 4, \tfrac{1}{4}, \tfrac{1}{4})\, x_2 \\ x \in D = \{x \in \mathbb{Z}^2 \mid -5x_1 + 4x_2 \leq 6\,;\, 3x_1 + x_2 \leq 9\,;\, x \geq 0\} \end{cases} \tag{6.1}$$

and let $(\tilde{P}_E)$ be the following main problem:

$$(\tilde{P}_E) \begin{cases} \max \tilde{\phi}(x) = (-4, -3, 1, \tfrac{1}{2})x_1 + (1, 2, \tfrac{3}{2}, 1)x_2 \\ x \in E(\tilde{P}) \end{cases}. \tag{6.2}$$

Suppose we want to get a satisfactory solution for $(\tilde{P}_{E_\pi})$:

**Iteration 1.**

**Step 1.1.** (Initialization) Set $\phi^{\text{opt}} = -\infty$, $\varepsilon = 0.01$, $\mathfrak{L}_1^0 = -1$, $\mathfrak{L}_2^0 = 0$, $\mathfrak{L}_1^1 = -5$, $\mathfrak{L}_2^1 = -3$.
Solve the relative deterministic relaxed problem $(P_0) : \max\{\phi^R(x) = -3x_1 + 2x_2 \,|\, x \in D\}$.
The obtained optimal solution is $x^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $\phi^0 = 2$ (see Fig. 6).

**Step 1.2.** (Testing $\pi$-efficiency of $x^0$)
The obtained optimal solution for $(T^L_{\{x^0\}})$ is $s^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ with $\Theta^L = 2.5 \neq 0$. And that obtained for $(T^R_{\{x^0\}})$ is $s^2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ with $\Theta^R = 17.75 \neq 0$. Thus, $x^0$ is not $\pi$-efficient but $s^1$ and $s^2$ are $\pi$-efficient (Properties 4.10 and 4.11).

**Step 1.3.** (Searching for equivalent admissible solutions eventually better than $s^1$ and $s^2$ with respect to $\phi^R$)
Solve $(Q_{\{s^1\}}) : \max\{\phi^R(x) = -3x_1 + 2x_2 \,|\, x \in D,\ \tilde{f}_1(x) = \tilde{f}_1(s^1),\ \tilde{f}_2(x) = \tilde{f}_2(s^1)\}$.
The obtained optimal solution is $x^1_{eq} = s^1$ and $\phi^1_{eq} = -2$.
Solve $(Q_{\{s^2\}}) : \max\{\phi^R(x) = -3x_1 + 2x_2 \,|\, x \in D,\ \tilde{f}_1(x) = \tilde{f}_1(s^2),\ \tilde{f}_2(x) = \tilde{f}_2(s^2)\}$.
The obtained optimal solution is $x^2_{eq} = s^2$ and $\phi^2_{eq} = 0$.

**Step 1.4.** Set $x^{\text{opt}} = x^2_{eq} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\phi^{\text{opt}} = \phi^2_{eq} = 0$. Reduce the search domain:

$$\begin{aligned} &D_1 = \{x \in D \,|\, 4.25x_1 - x_2 - 8.01\lambda_1^1 \geq -2\,;\, 1.25x_1 + 4.25x_2 - 10.01\lambda_2^1 \geq 0\,; \\ &2x_1 - 3x_2 - 3.51\gamma_1^1 \geq -6\,;\, -x_1 + 3x_2 - 6.51\gamma_2^1 \geq -3\,; \\ &4.25x_1 - x_2 - 7.01\lambda_1^2 \geq -2\,;\, 1.25x_1 + 4.25x_2 - 14.01\lambda_2^2 \geq 0\,; \\ &2x_1 - 3x_2 - 0.26\gamma_1^2 \geq -6\,;\, -x_1 + 3x_2 - 9.26\gamma_2^2 \geq -3\,; \\ &\lambda_1^1 + \lambda_2^1 + \gamma_1^1 + \gamma_2^1 \geq 1\,;\, \lambda_1^2 + \lambda_2^2 + \gamma_1^2 + \gamma_2^2 \geq 1\,; \\ &\lambda_i^1,\, \lambda_i^2,\, \gamma_i^1,\, \gamma_i^2 \in \{0, 1\}^4, \forall i = 1, 2\}. \end{aligned}$$

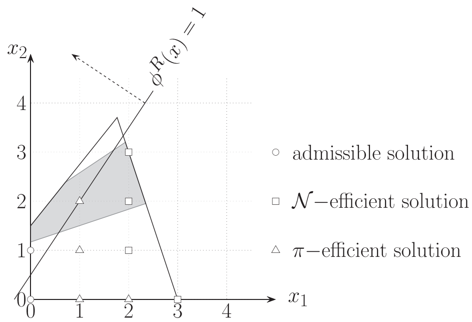


FIGURE 7. The search domain is $D_1$ in gray color and the optimal solution is achieved

Solving $(P_1) : \max\{\phi^R(x) = -3x_1 + 2x_2 \mid x \in D_1\}$ provides the optimal solution $x^1 = \begin{pmatrix}1\\2\end{pmatrix}$, $\lambda^1 = \lambda^2 = \begin{pmatrix}0\\0\end{pmatrix}$, $\gamma^1 = \begin{pmatrix}0\\1\end{pmatrix}$, $\gamma^2 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\phi^1 = 2$ (see Fig. 7). **CONTINUE.**

**Iteration 2.**

**Step 2.1.** (Testing $\pi$-efficiency of $x^1$)
$(T^L_{\{x^1\}})$ is unfeasible, then $x^1$ is $\pi$-efficient. **STOP.**
$x^{\text{opt}} = x^1 = \begin{pmatrix}1\\2\end{pmatrix}$ is a satisfactory solution of $(\tilde{P}_{E_\pi})$ and the relative objective fuzzy value is $\tilde{\phi}^{\text{opt}} = (-2, 1, 4, \frac{5}{2})$.

## 7. Numerical results

The method was programmed under Matlab14 using the CPLEX (version12.2) for Matlab toolbox function (cplexmilp). The instances were randomly generated and the program was run under windows10 installed on a PC with Intel(R) CORE(TM)i7-2.20 GHz Processor and 8.00 Gb of RAM.

The computational results are summarized in Tables 2 and 3 that give the sizes $(p \times m \times n)$ of instances and the corresponding minimum, median and maximum CPU time (in seconds). They give also the corresponding minimum median and maximum number of iterations. Computing minimum, median and maximum was performed on samples of

TABLE 2. Numerical results: Optimizing over a $\pi$-efficient set.

| | $p = 3$ | | $p = 6$ | | $p = 8$ | |
|---|---|---|---|---|---|---|
| $m \times n$ | CPU (s) | #*iter* | CPU (s) | #*iter* | CPU (s) | #*iter* |
| | min = 0.05 | min = 1 | min = 0.35 | min = 1 | min = 0.05 | min = 1 |
| $30 \times 60$ | med = 0.66 | med = 2.5 | med = 3.31 | med = 2 | med = 6.72 | med = 2 |
| | max = 1.83 | max = 7 | max = 10.20 | max = 4 | max = 76.70 | max = 4 |
| | min = 0.17 | min = 1 | min = 0.05 | min = 1 | min = 0.05 | min = 1 |
| $60 \times 90$ | med = 1.90 | med = 3.5 | med = 14.81 | med = 2.5 | med = 76.41 | med = 3 |
| | max = 7.28 | max = 9 | max = 41.68 | max = 8 | max = 230.45 | max = 5 |
| | min = 0.84 | min = 1 | min = 7.62 | min = 1 | min = 2.12 | min = 1 |
| $90 \times 120$ | med = 4.85 | med = 3.5 | med = 39.44 | med = 3.5 | med = 85.08 | med = 3.5 |
| | max = 17.02 | max = 6 | max = 110.21 | max = 6 | max = 541.49 | max = 4 |
| | min = 0.27 | min = 1 | min = 3.16 | min = 1 | min = 0.08 | min = 1 |
| $120 \times 120$ | med = 3.88 | med = 3 | med = 34.69 | med = 4 | med = 96.13 | med = 2 |
| | max = 11.22 | max = 7 | max = 103.99 | max = 7 | max = 715.79 | max = 5 |

TABLE 3. Numerical results: Optimizing over an $\mathcal{N}$-efficient set.

| | $p = 3$ | | $p = 6$ | | $p = 8$ | |
|---|---|---|---|---|---|---|
| $m \times n$ | CPU (s) | $\#iter$ | CPU (s) | $\#iter$ | CPU (s) | $\#iter$ |
| | min = 0.08 | min = 1 | min = 0.37 | min = 1 | min = 0.07 | min = 1 |
| $30 \times 60$ | med = 0.61 | med = 2.5 | med = 3.36 | med = 2.5 | med = 3.75 | med = 2 |
| | max = 1.75 | max = 6 | max = 15.80 | max = 5 | max = 66.10 | max = 4 |
| | min = 0.18 | min = 1 | min = 0.05 | min = 1 | min = 0.05 | min = 1 |
| $60 \times 90$ | med = 3.00 | med = 5.5 | med = 18.40 | med = 3.5 | med = 52.30 | med = 2.5 |
| | max = 12.91 | max = 8 | max = 36.97 | max = 6 | max = 197.15 | max = 6 |
| | min = 0.96 | min = 1 | min = 0.13 | min = 1 | min = 0.13 | min = 1 |
| $90 \times 120$ | med = 2.52 | med = 3 | med = 22.49 | med = 3 | med = 75.65 | med = 3 |
| | max = 10.23 | max = 4 | max = 87.99 | max = 9 | max = 703.80 | max = 4 |
| | min = 0.28 | min = 1 | min = 1.97 | min = 1 | min = 0.08 | min = 1 |
| $120 \times 120$ | med = 6.57 | med = 4.5 | med = 24.32 | med = 2 | med = 86.33 | med = 2 |
| | max = 9.76 | max = 6 | max = 130.78 | max = 7 | max = 690.53 | max = 4 |

10 instances of equal dimensions. Note that, in this implementation, we give medians instead of averages. As known, the advantage of use of the median is that it's much less affected by outlier values than the average would be.

Generating instances was performed as follows:

Elements of the fuzzy matrix $\tilde{A}$ were constructed to be positive in order to avoid the unboundedness of the admissible set. $\tilde{A}$ is input as a juxtaposition of four matrices $A_1$, $A_2$, $A_3$ and $A_4$. Elements of $A_1$ and $A_4$ were randomly generated following the uniform distribution on $[1;5]$ and they represent left and right spreads of those of $\tilde{A}$ respectively. Elements of $A_2$ were randomly generated following the uniform distribution on $[5;20]$ and they represent lower modal values of those of $\tilde{A}$. $A_3$ gives the upper modal values: $A_3 = A_2 + H$, where elements of $H$ represent the kernel's amplitudes and they were randomly generated following the uniform distribution on $[1;5]$. The fuzzy matrix $\tilde{C}$ and fuzzy vectors $\tilde{b}$ and $\tilde{d}$ were constructed in a similar way. Elements of $\tilde{C}$ and those of $\tilde{d}$ have kernel's values randomly generated between $-10$ and 10.

The computational results listed in Tables 2 and 3 show that the algorithm is effective for finding both optimal solutions: $\pi$-efficient and $\mathcal{N}$-efficient solutions. CPU time taken to solve each of the generated problems is essentially related to their sizes. In contrast, it seems that the number of iterations is not sensibly affected by the dimension of these problems.

## 8. Conclusion

The interest in Fuzzy Multiple Objective Linear Programming follows from the need of its application to solve several ambiguous operations research problems where parameters neither are ordinary numbers, nor its behaviors can be modeled by known probability distributions. To modeling imprecise information in these application domains it seems that Zadeh's fuzzy sets theory has a successful use. However, further study remain to be required in this field, since fuzzy numbers do not satisfy all properties that ordinary numbers have. Namely, equalities $\tilde{N} \div \tilde{N} = \tilde{1}$ and $\tilde{N} \ominus \tilde{N} = \tilde{0}$ do not hold in most cases, which makes it not an easy task to rank fuzzy numbers and there is no way to solve a fuzzy (single or multiple) linear programming program without transforming it into a deterministic one. The method presented in this paper is based on possibility and necessity measures for defining fuzzy efficient solutions namely $\pi$-efficient and $\mathcal{N}$-efficient solutions, and also for searching for an optimal solution of a fuzzy linear function among the efficient ones of a Fuzzy Multiple Objective Integer Linear Programming problem.

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