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Discrete Optimization

Optimizing a linear function over an integer efficient set

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Abstract

In this paper, a method for optimizing a linear function over the integer Pareto-optimal set without having to determine all integer efficient solutions is presented. The proposed algorithm is based on a simple selection technique that improves the linear objective value at each iteration. Two types of cuts are performed successively until the optimal value is obtained and the current truncated region contains no integer feasible solution.

Keywords: Multi-objective optimization; Discrete optimization; Linear programming; Efficient set

1. Introduction

Integer linear programming problems involving simultaneous conflicting objective functions have received considerable attention from several researchers and the literature in this area is extensive: see for instance [1,5,10,16,17,24]. Solving such problems amounts to finding all efficient (non-dominated) solutions.

The techniques used for solving multiple objective integer linear programming (MOILP) problems are diverse: cutting plane techniques, dual simplex procedures, branch and bound algorithms, dynamic programming approaches or iterative techniques that consist of solving a sequence of progressively more and more constrained single objective problems.

In practical applications of multiple criteria decision making, the decision makers often have to choose some preferred point from the efficient set E. This involves the problem of finding efficient solutions and describing the structure of E. Since, in many cases, the criteria are conflicting, the decision makers try to optimize one compromise criterion—possibly a linear one—over the efficient set, which raises the problem of finding a method for optimizing a function over the efficient set.

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In this paper, we focus on the problem of optimizing a *linear* function, denoted w, over the efficient set of a MOILP problem. We address the general case where w is any linear function and not necessarily a linear combination of the objective functions of the MOILP problem. A direct approach could consist of finding all efficient solutions of the MOILP problem and then finding the best value of w on that set. In view of the difficulty of determining the set of all efficient solutions, this approach is not appropriate for practical purposes. We thus propose an implicit technique that avoids searching for all efficient solutions but guarantees finding one that maximizes w.

A similar problem in the continuous case (i.e. optimizing w on the set of efficient solutions of a multiple objective linear programming (MOLP) problem has been tackled in [4,7,8,19,21].

For the discrete case, the only method we are aware of is that of Nguyen [11] which only gives an upper bound for the optimal objective value of w.

There are several reasons for studying such problems. For instance, Benson [3] describes a problem in which a manufacturer of four different types of products has ten factories. The key performance measures for a production plan $x \in S$ are profits $a(x)$ and employment levels $\{f^q(x) \mid 1 \leqslant q \leqslant 10\}$ at each of the ten factories. Profit is the main performance measure but a plan x is unacceptable if there exists one, y, in which employment levels are at least as good for y as for x, for each factory. One thus has to optimize the profit on the set of solutions that are also efficient from the employment point of view.

Another illustration can be encountered in combinatorial optimization (see [13]), where the minimum maximal flow problem is modelled as an optimization of a linear function over the efficient set.

Let $(V, s, t, E, c, \partial^+, \partial^-)$ denote a network the node set of which is V, with two designated nodes, source s and sink t, arc set E, non-negative capacity c_h for each arc h, incidence functions ∂^+ and ∂^- where $\partial^+ h$ is the node that arc h leaves and $\partial^- h$ is the node that arc h enters. A vector $x = (\ldots, x_h, \ldots)$ of dimension equal to $|E|$ is said to be a feasible flow if it satisfies the conservation equations and capacity constraints $\sum_{\{h|\partial^+ h=i\}} x_h = \sum_{\{h|\partial^- h=i\}} x_h$ for all nodes $i \in V\backslash\{s, t\}$ and $0 \leqslant x_h \leqslant c_h$ for all $h \in E$. The incidence matrix is therefore defined as follows: $A = (a_{ih})_{i\in V\backslash\{s,t\},h\in E}$ with

$$a_{ih} = \begin{cases} +1 & \text{if} \quad \partial^+ h = i, \\ -1 & \text{if} \quad \partial^- h = i, \\ 0 & \text{otherwise.} \end{cases}$$

The conservation equation becomes $Ax = 0$. The flow value $\phi(x)$ of a feasible flow x is given by $\phi(x) = \sum_{\{h|\partial^+ h=s\}} x_h - \sum_{\{h|\partial^- h=s\}} x_h$.

The problem stated above can be formulated as

$$\begin{cases} \text{minimize} & \phi(x) \\ \text{s.t.} & x \in X_E, \end{cases}$$

where $\phi(x) = dx$ is a linear function of the flow x and X_E is the efficient set of the multiple objective linear programming problem defined as follows:

$$\begin{cases} \text{maximize} & Ix \\ \text{s.t.} & x \in X, \end{cases}$$

where I is the identity matrix of dimension $|E|$, and X is the set of feasible flows defined by $X = \{x | x \in \mathbb{R}^{|E|};\ Ax = 0;\ 0 \leqslant x \leqslant c\}$.

The algorithm that we propose is inspired by the work of Ecker and Song [7] and Benson and Sayin [4] for the continuous case. But, as is generally the case, passing from MOLP to MOILP is not trivial. For instance, as observed in [16], when seeking to generate the efficient set of MOILP, the non-supported efficient solutions can be missed when applying the so called Geoffrion's principle (i.e. maximizing a weighted sum of the criteria over the feasible set while letting the weights vary), see [16]. Since the methods used in

[4,7] are not valid in our case, we mainly keep the idea of choosing the best direction for improving w at each iteration. To be more specific let us introduce precisely the problem to be solved.

Mathematically, the MOILP problem is described as the problem of finding all efficient solutions of

$$(P) \quad \begin{cases} \max & Z_i = C^i X, \quad i = 1, 2, \ldots, p \\ \text{s.t.} & X \in S, \end{cases}$$

where $S = D \cap \mathbb{Z}^n$, $D = \{X \in \mathbb{R}^n | AX = b, X \geqslant 0\}$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $p \geqslant 2$; $C^1, C^2, \ldots, C^p \in \mathbb{Q}^n$ are row vectors, $\mathbb{Z}$ is the set of integers and $\mathbb{Q}$ is the set of rational numbers. We assume throughout the paper that S is not empty and D is a bounded convex polyhedron. The set of all integer efficient solutions of (P) is denoted by $E(P)$. Efficiency and non-dominance are defined as follows (see [14,22,23]):

Definition 1. A point $\overline{X} \in S$ is an *efficient* solution if and only if there is no $X \in S$ such that $Z_i(X) \geqslant Z_i(\overline{X})$ for all $i \in \mathfrak{I} = \{1, 2, \ldots, p\}$ and $Z_i(X) > Z_i(\overline{X})$ for at least one $i \in \mathfrak{I}$. Otherwise, $\overline{X}$ is not efficient and $(Z_1(\overline{X}), Z_2(\overline{X}), \ldots, Z_p(\overline{X}))$ is said to be *dominated.*

The central problem that we are studying is:

$$(P_E) \quad \begin{cases} \max & w = dX \\ \text{s.t.} & X \in E(P), \end{cases}$$

where d denotes an n dimensional row vector and the jth component of which, d_j, is a rational number.

Let the relaxed problem be:

$$(P_R) \quad \begin{cases} \max & w = dX \\ \text{s.t.} & X \in S \end{cases}$$

and define the problem $(P_i(S))$, $i \in \{1, 2, \ldots, p\}$ by

$$(P_i(S)) \quad \begin{cases} \max & Z_i = C^i X \\ \text{s.t.} & X \in S. \end{cases}$$

We define the problem $(P_i(D))$, $i \in \{1, 2, \ldots, p\}$ by

$$(P_i(D)) \quad \begin{cases} \max & Z_i = C^i X \\ \text{s.t.} & X \in D. \end{cases}$$

It may happen that the optimal solution X^0 of problem $(P_i(S))$ is not unique. In this case, there is another feasible solution $X^1 \neq X^0$ with $Z_i(X^1) = Z_i(X^0)$. We refer to X^1 as an *alternate* optimal solution of $(P_i(S))$.

A naive way of solving problem (P_E) is to build the set $E(P)$ of all efficient solutions of (P) and then to optimize $w = dX$ on that set. Our method avoids enumerating explicitly all efficient solutions of (P). The detailed presentation of the algorithm is given in Sections 3 and 4. In Section 2 we introduce the notation and establish a number of theoretical results that will help justifying the algorithm in Section 3. A flowchart of the algorithm is produced in the Appendix A.

In Section 5, we illustrate by a numerical example, how the algorithm works. Some conclusions are delivered in Section 6.

2. Notation and preliminary results

Beside the notation already introduced for describing the problems (P) and (P_E), we also use the notation introduced by Gupta and Malhotra [9] and Verma [18]:

- $Z_1, Z_2, \ldots, Z_p$ denote the criteria and w, the additional criterion;
- D_1 is the set $\{X \in \mathbb{R}^{n_1} | A_1X = b_1, A_1 \in \mathbb{Q}^{m_1 \times n_1}, b_1 \in \mathbb{Q}^{m_1}, (m_1, n_1) \in \mathbb{N} \times \mathbb{N}, m_1 \neq 0, n_1 \neq 0, X \geqslant 0\}$, which is the current truncated region of D obtained by successive Gomory cuts introduced when optimizing problem $(P_1(S))$; $S_1 = D_1 \bigcap \mathbb{Z}^{n_1}$. Note that $S_1 = S = D \bigcap \mathbb{Z}^{n_1}$, because Gomory cuts do not eliminate integer solutions from D.
- $(Z_1^1, Z_2^1, \ldots, Z_p^1)$ is the first non-dominated p-tuple corresponding to the optimal integer solution X_1 obtained in D_1, where $Z_i^1 = C^iX_1$, for $i = 1, 2, \ldots, p$.

For $k \geqslant 1$, we have:

- $X_k \in \mathbb{Z}^{n_k}$ is one optimal integer solution obtained in D_k (see below);
- B_k is the basis associated with solution X_k;
- $a_{k,j} \in \mathbb{Q}^{m_k \times 1}$ is the activity vector of $x_{k,j}$ with respect to the current truncated region D_k;
- $I_k = \{j \mid \text{the vector } a_{k,j} \text{ is a column of the basis } B_k\}$ (indices of basic variables);
- $N_k = \{j \mid \text{the vector } a_{k,j} \text{ is not a column of the basis } B_k\}$ (indices of non-basic variables);
- $y_{k,j} = (y_{k,ij}) = (B_k)^{-1}a_{k,j}$, where $y_{k,j} \in \mathbb{Q}^{m_k \times 1}$;
- $\Gamma_k = \{j \in N_k | z_{1,j}^k - c_j^1 \geqslant 0 \text{ and } w_j^k - d_j^k \leqslant 0\}$, where $z_{1,j}^k = C_{B_k}^1 y_{k,j}$, $C_{B_k}^1$ is the vector of cost coefficients of basic variables associated with B_k in vector C^1 and c_j^1 is the jth component of vector C^1; $w_j^k = d_{B_k} y_{k,j}$, with d_{B_k} the vector of cost-coefficients of basic variables associated with B_k in vector d;
- $D_k = \{X \in \mathbb{R}^{n_k} | A_kX = b_k, A_k \in \mathbb{Q}^{m_k \times n_k}, b_k \in \mathbb{Q}^{m_k}, (m_k, n_k) \in \mathbb{N} \times \mathbb{N}, m_k \neq 0, n_k \neq 0, X \geqslant 0\}$ for $k \geqslant 2$, where D_k is the current truncated region obtained after having applied the cut $\sum_{j \in N_{k-1} \setminus \{j_{k-1}\}} x_j \geqslant 1$, with $j_{k-1} \in \Gamma_{k-1}$, or the cut $dX \geqslant dX_k$ and successive Gomory cuts if necessary in each of these cases; $S_k = D_k \bigcap \mathbb{Z}^{n_k}$.

Table 1 shows a tableau of the type that is used when applying the simplex or the dual simplex procedure. It differs from the classical one in the bottom lines. There are all together $p + 1$ lines corresponding to the criteria $Z_1, \ldots, Z_p$ and w; in each line, one finds the reduced cost $z_{i,j}^k - c_j^i$ associated to the corresponding criterion. In table 1, we have

$$t_{ij} = \begin{cases} y_{k,ij} & \text{if } j \in N_k, \\ e_j & \text{if } j \in I_k; \end{cases}$$

e_j is the jth column of the identity matrix of dimension $m_k \times m_k$.

Table 1
Generic simplex tableau

Tableau	Value of basic variable	x_1	x_2	...	x_{n_k}
X_{B_k}	$x_{k,1}$	t_{11}	t_{12}	...	t_{1n_k}
	$x_{k,2}$	t_{21}	t_{22}	...	t_{2n_k}
	...	...	...	...	....
	x_{k,m_k}	t_{m_k1}	t_{m_k2}	...	$t_{m_kn_k}$
Reduced cost	Z_1^k	$z_{1,1}^k - c_1^1$	$z_{1,2}^k - c_2^1$	...	$z_{1,n_k}^k - c_{n_k}^1$
	...	...	...	....	...
	Z_p^k	$z_{p,1}^k - c_1^p$	$z_{p,2}^k - c_2^p$	...	$z_{p,n_k}^k - c_{n_k}^p$
	w^k	$w_1^k - d_1^k$	$w_2^k - d_2^k$	...	$w_{n_k}^k - d_{n_k}^k$

2.1. Gomory cuts

In the algorithm, we use Gomory's fractional cutting plane technique (see [20, p. 124]) that we recall briefly below.

Consider an integer linear program like $(P_1(S))$. First we solve the associated linear programming relaxation of $(P_1(D))$, say,

$$(P_1(D)) \quad \begin{cases} \max & Z_1 = C^1X \\ \text{s.t.} & X \in D. \end{cases}$$

Consider an optimal basis if it exists; we denote by I and N respectively the basic index set and the non-basic index set. We choose a basic variable that is not an integer.

Let $x_i = t_{i_0}$ be the righthand side of the ith constraint, $i \in I$. The corresponding constraint $x_i + \sum_{j\in N} t_{ij}x_j = t_{i0}$ is $x_i = [t_{i0}] + f_{i0} - \sum_{j\in N}[t_{ij}]x_j - \sum_{j\in N} f_{ij}x_j$ (using the notation $t = [t] + f$, where $[t]$ is the integer part of t and f is the fractional part of t; $0 \leqslant f \leqslant 1$). Under the necessary conditions that x_i and $x_j, j \in N$, are integers, the Gomory cut is $s_i = -f_{i0} + \sum_{j\in N} f_{ij}x_j$, where the slack variable s_i is a non-negative integer variable. This constraint is introduced in the simplex tableau and the problem is solved using the dual simplex method. After a finite number of iterations, we either obtain an optimal integer solution or we find that the problem is infeasible.

2.2. Testing efficiency

The following result (see [2, Theorem 3.1]) is used in various steps of the algorithm to test the efficiency of a given feasible solution of problem (P).

Theorem 1. *Let X^* be an arbitrary element of the region S. $X^* \in E(P)$ if and only if the optimal value of the objective V is null in the following mixed integer linear programming problem*:

$$(P(X^*)) \quad \begin{cases} \max & V = \sum_{i=1}^{p} \psi_i \\ \text{s.t.} & \begin{cases} CX = I\Psi + CX^*, \\ X \in S; \quad \psi_i \text{ are real non-negative variables for } i = 1, 2, \ldots, p, \end{cases} \end{cases}$$

where C is the $m \times n$ matrix, the ith row of which is C^i, $i = 1, 2, \ldots, p$, I is the identity matrix $(p \times p)$ and $\Psi = (\psi_i)_{i=1,\ldots,p}$.

2.3. Cuts of type I

The algorithm we propose is based on exploring the edges incident to a solution and cutting edges instead of solutions.

Definition 2. Assume that $j_k \in N_k$. An edge E_{j_k} incident to a solution X_k is defined as the set

$$E_{j_k} = \left\{ (x_i) \in D_k \left| \begin{array}{l} x_i = x_{k,i} - \theta_{j_k} y_{k,ij_k} \text{ for } i \in I_k \\ x_{j_k} = \theta_{j_k} \\ x_\alpha = 0, \quad \text{for all } \alpha \in N_k \setminus \{j_k\} \end{array} \right. \right\},$$

where $0 < \theta_{j_k} \leqslant \min_{i\in I_k}\left\{\frac{x_{k,i}}{y_{k,ij_k}} | y_{k,ij_k} > 0\right\}$, θ_{j_k} is a positive integer and $\theta_{j_k} \times y_{k,ij_k}$ are integers for all $i \in I_k$ if such integer values exist.

Note that in our definition X_k does not belong to the edge E_{j_k}.

We present some results that will support the fact that the procedure terminates. The following theorem addresses the case in which the optimal solution of $(P_1(S))$ is not unique. Note that a sufficient condition for the uniqueness of the optimal solution X_1 of $(P_1(S))$ is that the set $J_1 = \{j \in N_1 | z^1_{1,j} - c^1_j = 0\}$ is empty.

Theorem 2. *Let X_1 be an optimal solution of problem $(P_1(S))$. All integer feasible solutions of problem $(P_1(S))$ alternate to X_1 on an edge E_{j_1} of region D (or truncated region D_1) emanating from it, in the direction of a vector a_{1,j_1}, $j_1 \in J_1$ with $J_1 = \{j \in N_1 | z^1_{1,j} - c^1_j = 0\}$, lie in the open half space $\sum_{j \in N_1 \setminus \{j_1\}} x_j < 1$.*

Proof. Let X_1 be an optimal solution of $(P_1(D))$. $A_1X_1 = \sum_{i \in I_1} a_{1,i}x_{1,i} = b$.

Let $j_1 \in J_1$; we have $\sum_{i \in I_1} a_{1,i}x_{1,i} - \theta_{j_1}a_{1,j_1} + \theta_{j_1}a_{1,j_1} = b$, where θ_{j_1} is a non-zero positive scalar. Trivially, $a_{1,j_1} = \sum_{i \in I_1} a_{1,i}y_{1,ij_1}$; hence:

$$\sum_{i \in I_1} a_{1,i}x_{1,i} - \theta_{j_1}\left(\sum_{i \in I_1} a_{1,i}y_{1,ij_1}\right) + \theta_{j_1}a_{1,j_1} = b;$$
$$\sum_{i \in I_1} a_{1,i}(x_{1,i} - \theta_{j_1}y_{1,ij_1}) + \theta_{j_1}a_{1,j_1} = b.$$

For $0 < \theta_{j_1} \leqslant \min_{i \in I_1}\left\{\frac{x_{1,i}}{y_{1,ij_1}} | y_{1,ij_1} > 0\right\}$, we define X_2 as follows:

$$X_2 = \begin{cases} x_{2,i} = x_{1,i} - \theta_{j_1} \times y_{1,ij_1}, & i \in I_1, \\ x_{2,j_1} = \theta_{j_1}, & \\ x_{2,\alpha} = 0, & \text{for all } \alpha \in N_1 \setminus \{j_1\}, \end{cases}$$

which is a new integer feasible solution of $(P_1(S))$, provided that θ_{j_1} is a positive integer and $\theta_{j_1} \times y_{1,ij_1}$ are integers for all $i \in I_1$.

We now show that $Z_1(X_2) = Z_1(X_1)$.

$$\begin{aligned} Z_1(X_2) = C^1X_2 &= \sum_{i \in I_1} c^1_i x_{2,i} + c^1_{j_1}x_{2,j_1} + \sum_{\alpha \in N_1 \setminus \{j_1\}} c^1_\alpha x_{2,\alpha} \\ &= \sum_{i \in I_1} c^1_i(x_{1,i} - \theta_{j_1}y_{1,ij_1}) + c^1_{j_1}\theta_{j_1} = \sum_{i \in I_1} c^1_i x_{1i} - \sum_{i \in I_1} c^1_i\theta_{j_1}y_{1,ij_1} + c^1_{j_1}\theta_{j_1} \\ &= \sum_{i \in I_1} c^1_i x_{1i} - \theta_{j_1}\left(\sum_{i \in I_1} c^1_i y_{1,ij_1} - c^1_{j_1}\right) = Z_1(X_1) - \theta_{j_1}(z^1_{1,j} - c^1_{j_1}). \end{aligned}$$

As $j_1 \in J_1$, then $z^1_{1,j} - c^1_{j_1} = 0$. Thus $Z_1(X_2) = Z_1(X_1)$.

X_2 is an integer feasible solution of $(P_1(S))$, alternate to X_1, lying on an edge

$$E_{j_1} = \left\{ (x_i) \in \mathbb{R}^{(|I_1|+|N_1|)} \middle| \begin{array}{l} x_{2,i} = x_{1,i} - \theta_{j_1} \times y_{1,ij_1}, \quad i \in I_1 \\ x_{2,j_1} = \theta_{j_1} \\ x_{2,\alpha} = 0, \quad \text{for all } \alpha \in N_1 \setminus \{j_1\} \end{array} \right\}. \tag{1}$$

We have $\sum_{j \in N_1 \setminus \{j_1\}} x_{2,j} < 1$, since $x_{2,j} = 0$ for all $j \in N_1 \setminus \{j_1\}$. Thus, the point X_2 lies in the open half space $\sum_{j \in N_1 \setminus \{j_1\}} x_j < 1$. □

Eq. (1) enables us to compute the integer feasible alternate solutions when the optimal solution obtained by solving $(P_1(S))$ is not unique.

The following theorem suggests a cut that can be viewed as a generalization of Dantzig's cut (see [15, p. 178] and [6]); it truncates a whole edge while the latter truncates only a point. Obviously, it leads to a reduction of the feasible set that is more drastic than the classical Dantzig cut.

Theorem 3. *An integer feasible solution of problem $(P_1(S_k))$ that is distinct from X_k and not on an edge E_{j_k} of the truncated region D_k (or region D) through an integer optimal solution X_k of $(P_1(S))$ lies in the closed half space*

$$\sum_{j\in N_k\setminus\{j_k\}} x_j \geqslant 1. \tag{2}$$

Proof (by contradiction). Let $\widetilde{X} = (\widetilde{x}_i)$ be an integer feasible solution of problem $(P_1(S))$ not on an edge E_{j_k} such that it does not satisfy (2). We will prove it is impossible. This assumption implies (Theorem 2) that $\widetilde{x}_j = 0$, for all $j \in N_k\setminus\{j_k\}$. There are two cases to consider: either $\widetilde{x}_{j_k} > 0$ or $\widetilde{x}_{j_k} = 0$.

Case 1. $\widetilde{x}_{j_k} > 0$.

- If $\widetilde{x}_{j_k} > \min_{i\in I_k}\left\{\frac{x_{k,i}}{y_{k,ij_k}}\middle| y_{k,ij_k} > 0\right\} = \frac{x_{k,q}}{y_{k,qj_k}}$ (say), then $\widetilde{x}_{k,q} = x_{k,q} - \widetilde{x}_{j_k} y_{k,qj_k} < 0$, which makes the solution infeasible.
- If $\widetilde{x}_{j_k} \leqslant \min_{i\in I_k}\left\{\frac{x_{k,i}}{y_{k,ij_k}}\middle| y_{k,ij_k} > 0\right\}$, then $\widetilde{x}_{j_k} \leqslant \frac{x_{k,i}}{y_{k,ij_k}} \forall i \in I_k$, and it is easy to show that, $\widetilde{x}_{k,i} = x_{k,i} - \widetilde{x}_{j_k} y_{k,ij_k}$, which implies that $\widetilde{X}$ lies on the edge E_{j_k}, contrary to the hypothesis.

Case 2. $\widetilde{x}_{j_k} = 0$, then the index sets of basic and non-basic variables in the optimal tableau corresponding to $\widetilde{X}$ are respectively B_k and N_k, and therefore $\widetilde{X} = X_k$, contrary to the hypothesis.

Each of these cases leads to a contradiction. So the initial assumption ($\widetilde{X}$ does not satisfy the inequality (2) must be false. Hence $\widetilde{x}_j > 0$ for at least one $j \in N_k\setminus\{j_k\}$ implying that $\widetilde{X}$ lies in the closed half space $\sum_{j\in N_k\setminus\{j_k\}} x_j \geqslant 1$. $\square$

Cut of type I. The inequality (2) introduced in this theorem will be called a cut of type I. It will be used in the method in order to cut off all integer feasible solutions on an edge incident to an optimal solution X_k of (P_1), including X_k itself, from the current feasible domain S_k.

Remark 4. Let X_k be an optimal solution of $(P_1(S_k))$ and suppose that Γ_k is empty (i.e. there is no edge like mentioned in Theorem 3). Then an integer feasible solution of $(P_1(S_k))$ distinct from X_k lies in the closed half space defined by the Dantzig cut

$$\sum_{j\in N_k} x_j \geqslant 1. \tag{3}$$

We now calculate the value w'_k of the linear function w at any solution $X'_k = (x'_1, x'_2, \ldots, x'_n)$ lying on E_{j_k}:

$$w'_k = \sum_{j=1}^{n} d_j^k x'_j = \sum_{i\in I_k} d_i^k (x_{k,i} - \theta \times y_{k,ij_k}) + d_{j_k}^k \times \theta = \left(d_{j_k}^k - \sum_{i\in I_k} d_i^k \times y_{k,ij_k}\right)\theta + \sum_{i\in I_k} d_i^k x_{k,i},$$

where θ is an integer verifying $0 < \theta \leqslant \theta_{j_k}^0$ and $\theta_{j_k}^0$ is the integer part of $\min_{i\in I_k}\left\{\frac{x_{k,i}}{y_{k,ij_k}}\middle| y_{k,ij_k} > 0\right\}$. We put:

$$\beta_k = \left(d_{j_k}^k - \sum_{i\in I_k} d_i^k \times y_{k,ij_k}\right). \tag{4}$$

Then along an edge $E_{j_k}; j_k \in \Gamma_k$, we have $\beta_k \geqslant 0$. Therefore, the values of w'_k are increasing and w'_k reaches its maximum for $\theta = \theta_{j_k}^0$.

Finally we recall a well known result (see [12]).

Corollary 5. *A point X^0 that is a unique solution of the integer linear programming problem*

$$(P_i(S))\begin{cases} \max & Z_i = C^i X \\ \text{s.t.} & X \in S, \end{cases}$$

where Z_i is any of the objective functions of problem (P), is efficient.

This corollary follows directly from the definition of efficiency. Of course, if X^0 is not unique, then it may be dominated by another optimal solution of (P_i).

3. Informal description of the procedure

This section describes informally but precisely the algorithm that will be further detailed in Section 4. We prove that the algorithm yields an optimal solution of (P_E) in a finite number of steps. Before starting the description we introduce another type of cut.

Cut of type II. The following inequality is called a cut of type II:

$$dX \geqslant w_{\text{opt}}. \tag{5}$$

Such cuts are imposed, at some occasions that will be made precise below, after X_{opt} and w_{opt} have been updated as a consequence of having found a new better efficient solution of (P).

3.1. Finding an efficient solution

Firstly, we search for a first efficient solution of (P). Set S_1 to S and D_1 to D. We start from X_1, an optimal solution of $(P_1(S_1))$. If it is efficient, it is a first efficient solution of (P) and we initialize $X_{\text{opt}} = X_1$ and $w_{\text{opt}} = dX_1$. If not, either Γ_1 is empty and we apply a Dantzig cut which reduces the domain D_1 or it is not and we explore the edges associated with Γ_1 searching for an efficient solution of (P). If we find one, say X'_i, on one of the edges, we can initialize X_{opt} and w_{opt}. Otherwise, we choose one of the edges (for instance the one that contains more integer feasible solutions of $(P_1(S_1))$ and we apply a cut of type I which reduces the domain D_1; we apply the dual simplex algorithm, and Gomory cuts if necessary, to obtain X_2, an optimal solution of $(P(S_2))$. We continue from X_2 as we did from X_1 and iteratively until an efficient solution is ultimately found. Suppose it is obtained at iteration r; the efficient solution can either be X_r (optimal solution of $(P_1(S_r))$ or a solution X'_r on one of the edges corresponding to Γ_r.

Proposition 6. *Under the hypothesis that S is not empty, and D bounded, the procedure ends up with an efficient solution of (P).*

Proof. Since D is bounded, S is non-empty and finite. Each cut of Dantzig or of type I reduces strictly the domain. Hence the procedure terminates with an efficient solution because at least one such solution exists in S. □

3.2. General iteration

We now describe a general iteration k, which is posterior to iteration r at which X_{opt} and w_{opt} were initialized (see Section 3.1).

D_k is the current feasible region; it has been obtained by imposing successively on D three types of cuts (plus, possibly, Gomory cuts); Dantzig cuts, cuts of type I (see after Theorem 3) and cuts of type II (see Eq. (5)). Solve the problem $(P_1(S_k))$. It differs from $(P_1(S_{k-1}))$, solved at the previous iteration, by the adjunction of a single constraint. We start from the previous simplex tableau and we use the dual simplex algorithm possibly with Gomory cuts. It ends up with one of the following conclusions:

- either the current feasible region is empty,
- or an optimal solution X_k of $(P_1(S_k))$ is found.

In the former case, the algorithm stops and outputs X_{opt} as an optimal solution of (P_E). If a new optimal solution X_k is found, there are two cases to be considered.

1. X_k is not efficient or X_k is efficient and $dX_k < w_{opt}$. If Γ_k is empty we apply a Dantzig cut which reduces the domain D_k. Otherwise, we start exploring all edges incident to X_k corresponding to Γ_k until an efficient solution improving w_{opt} is found.
   a. If no such an efficient solution is found after all edges are examined, choose one of these edges and apply a cut of type I which reduces D_k; this leads to iteration $k+1$.
   b. If an efficient solution X'_k improving on w_{opt} is found, stop the exploration of the edges, update X_{opt} and w_{opt} and apply a cut of type II: $dX \geqslant dX'_k$. This cut does not necessarily reduce the domain D_k (in case $dX_k = dX'_k$). Start iteration $k+1$.
2. In the second case, X_k is efficient and $dX_k \geqslant w_{opt}$. Either we apply a Dantzig cut (if $\Gamma = \emptyset$) or we explore the edges incident to X_k corresponding to Γ_k as above, but only those on which w strictly increases ($\beta > 0$), excluding those edges on which w remains constant ($\beta = 0$). Indeed, exploring the edges on which w remains constant can not bring us a new efficient solution better than X_{opt}. Except for this restriction on the edges to be considered, the search is done as in the previous case; it results in a Dantzig cut or a cut of type I, if no efficient solution better than X_k is found, or in a new efficient solution X'_k better than X_k and a cut of type II.

Proposition 7. *After an iteration k, with $k \geqslant r$, is completed, and provided the algorithm did not stop (i.e. a solution X_k was found), either the domain D_k is strictly reduced or the best value of w so far, w_{opt}, has strictly improved.*

Proof. We see from the description of the general iteration that, during iteration k, either a cut of type I or a Dantzig cut is applied (which strictly reduces the domain) or a new efficient solution is found, that improves w_{opt}. □

Note that both events listed in the proposition may occur; it happens when X_k is efficient, improves w_{opt} and the exploration of the edges incident to X_k does not yield a better efficient solution X'_k. We are now in position to prove the following theorem.

Theorem 8. *If S is non-empty and D is bounded, then*

(1) *the algorithm terminates in a finite number of iterations*;
(2) *the solution X_{opt} is an optimal solution of problem (P_E).*

Proof. Proposition 6 guarantees that we can obtain an initial efficient solution of (P), at iteration r, $r \geqslant 1$. By Proposition 7, we know that at each iteration k, with $k \geqslant r$, the domain is strictly reduced (by a Dantzig cut or a cut of type I) or w_{opt} strictly increases.

Obviously, since the domain S is finite, it may not be strictly reduced an infinite number of times. For the same reason, only a finite number of improvements of $w = dX$ may be observed when X moves in the finite set S. This proves that the algorithm stops after a finite number of iterations.

Provided S is non-empty and D is bounded, the algorithm stops at iteration $k > r$ if and only if the problem $(P_1(S_k))$ is infeasible; this is seen from the fact that, the dual simplex algorithm, at some stage, possibly after the adjunction of Gomory cuts, can not perform any pivot operation. The current value of w_{opt} at that iteration is optimal and X_{opt} is an optimal solution. This is due to the fact that

- Dantzig cuts only eliminate one solution, X_k, that has been explored;
- cuts of type I only eliminate edges that have been explored, i.e. on which there are no efficient integer solutions better than the current optimal solution;
- cuts of type II only eliminate solutions that are strictly worse than the current optimal solution;
- Gomory cuts do not eliminate integer solutions. $\square$

4. Technical presentation of the algorithm

We give a technical description of the algorithm presented in the previous section. An initial step is added, just to check whether, by chance, we could solve problem (P_E) just by solving the relaxed problem (P_R). The flowchart of this algorithm is in the Appendix.

Step 1. We solve the relaxed problem (P_R). Let X^* be an optimal solution. This solution is tested for efficiency by solving problem $(P(X^*))$ stated in Theorem 1 (see Section 2.2).

If it is efficient, then it is also a solution of (P_E) and the algorithm terminates. Otherwise, we go to Step 2.

Step 2. Let $w_{\text{opt}} = -\infty$. We solve the problem $(P_1(S))$. (One may alternatively consider any of the problems $(P_i(S);\ i = 2, 3, \ldots, p)$ instead of $(P_1(S))$.)

2.1. If $J_1 = \{j \in N_1 | z^1_{1,j} - c^1_j = 0\} = \emptyset$, then the optimal solution found, X_1, is unique and it is efficient (Corollary 5). Let $w^1 = dX_1$, set w_{opt} to w^1, set X_{opt} to X_1 and go to step 3.

2.2. If $J_1 \neq \emptyset$, then the optimal solution X_1 of problem $(P_1(S))$ may not be unique, test the efficiency of X_1 (Section 2.2); if it is not efficient go to step 3; otherwise let $w^1 = dX_1$, set w_{opt} to w^1, set X_{opt} to X_1, and go also to step 3.

Step 3. Let $k = 1$ and perform the following sub-steps:

3.1. Construct the set $\Gamma_k = \{j \in N_k | z^k_{1,j} - c^1_j \geqslant 0 \text{ and } w^k_j - d^k_j \leqslant 0\}$.

- If $\Gamma_k = \emptyset$, then go to step 3.2 and the cut in that step becomes the Dantzig cut $\sum_{j \in N_k} x_j \geqslant 1$.
- Otherwise, let $\gamma = \Gamma_k$. Go to (a).

(a) If $\gamma = \emptyset$, then let $j_k \in \Gamma_k$ and go to (sub-step 3.2). Otherwise, select $j_k \in \gamma$ and calculate $\theta^0_{j_k}$ the integer part of $\min_{i \in I_k} \left\{ \frac{x_{k,i}}{y_{k,ij_k}} \middle| y_{k,ij_k} > 0 \right\}$.

- If $\theta^0_{j_k} = 0$, then there is no integer feasible solution on the edge E_{j_k}, put $\gamma := \gamma \backslash \{j_k\}$ and go to (a).
- Otherwise, if $\theta^0_{j_k} \geqslant 1$, then go to (b).

(b) If X_k is efficient and $dX_k \geqslant w_{\text{opt}}$, then calculate the value of the parameter β_k defined in Eq. (4). If this value is not equal to zero, then go to (c), otherwise, put $\gamma := \gamma \backslash \{j_k\}$ and go to (a).If X_k is not efficient or $dX_k < w_{\text{opt}}$, then go to (c) (the edge E_{j_k} is explored regardless of the value of β_k).

(c) Explore the edge E_{j_k}, searching for a feasible solutions of $(P_1(S))$ corresponding to θ and test for efficiency starting from $\theta = \theta^0_{j_k}$ until $\theta = 1$ (θ is a positive integer). Once a first integer efficient solution is found, say X'_k, such that $dX' > w_{\text{opt}}$, set X_{opt} to X'_k and w_{opt} to dX'_k and go to sub-step 3.2. If there is no integer efficient solution on this edge, then put $\gamma := \gamma \backslash \{j_k\}$ and go to (a).

3.2. Let $k := k + 1$. Define the new truncated region D_k as the subset of D_{k-1} obtained by applying the cut $dX \geqslant dX'_{k-1}$ (cut of type II) and using the dual simplex method and Gomory cuts—whenever they are needed—to find a new optimal solution X_k. Set X_{opt} to X_k and w_{opt} to dX_k and go to sub-step 3.1.

3.3. Let $k := k + 1$. The new truncated region D_k is obtained as a subset of D_{k-1} (or D if $k = 1$) by applying the specified cut (Dantzig cut or cut of type I) and using the dual simplex method and Gomory cuts—whenever they are needed—to find a new optimal solution X_k; let $w^k = dX_k$.

Set the variable X_{opt} to X_k and w_{opt} to w^k if the solution X_k is efficient and $dX_k > w_{\text{opt}}$; go to sub-step 3.1, otherwise, go to sub-step 3.1.

Terminal Step. The procedure terminates either at the first step when the solution X^0 is efficient or the impossibility of pivot operations appears indicating that the current region contains no integer feasible point. The optimal solution is then X_{opt} and its value on criterion w is w_{opt}.

5. Numerical illustration

Consider the following example (Gupta [9])

$$(P)\begin{cases} \max & Z_1 = x_1 + 2x_2 \\ \max & Z_2 = 3x_1 - 2x_2 \\ \max & Z_3 = -x_1 + 2x_2 \\ \text{s.t.} & \begin{cases} x_1 + x_2 \leqslant 7, \\ 2x_1 \leqslant 11, \\ 2x_2 \leqslant 7, \\ x_1, x_2 \geqslant 0 \text{ and integers.} \end{cases} \end{cases}$$

Let the main problem be

$$(P_E)\begin{cases} \max & W = -2x_1 - 3x_2 \\ \text{s.t.} & x_1, x_2 \in E(P) \end{cases}$$

(see Fig. 1 below).

Step 1. The relaxed problem (P_R) is being solved. The optimal solution is $w^0 = 0$ for $X^* = (0,0)'$ which is not efficient. Go to step 2.

Tableau I

Basis	Value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6
x_1	4	1	0	1	0	0	−1
x_4	3	0	0	−2	1	0	2
x_2	3	0	1	0	0	0	1
x_5	1	0	0	0	0	1	−2
$Z^1_{1,j} - c^1_j$	10	0	0	1	0	0	1
$w^1_j - d^1_j$	−17	0	0	−2	0	0	−1

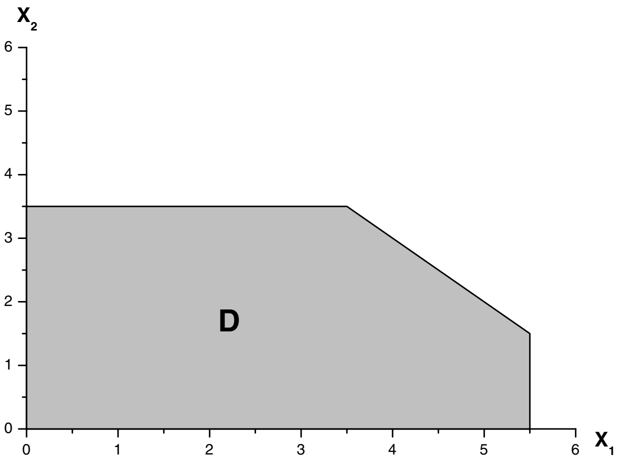

Fig. 1. The feasible region $S_1 = S = D \cap \mathbb{Z}^2$.

Step 2. We solve the problem $(P_1(S))$ and let $w_{\text{opt}} = -\infty$. The results of solving problem $(P_1(S))$ are summarized in Tableau I.

The optimal solution $X_1 = (4,3)'$ is unique thus it is efficient (Corollary 5). Let it be a first efficient solution that corresponds to $w^1 = -17$.

We have, $dX_1 = -17$, then $w_{\text{opt}} = -17$ and $X_{\text{opt}} = (4,3)'$.

$\mathbf{k} = \mathbf{1}$; $I_1 = \{1,2,4,5\}$, $N_1 = \{3,6\}$.

$\Gamma_1 = \{j \in N_1 | z^1_{1,j} - c^1_j \geqslant 0 \text{ and } w^1_j - d^1_j \leqslant 0\} = \{3,6\} \neq \emptyset$. We put $\gamma = \Gamma_1 = \{3,6\}$.

Let $j_1 = 3 \in \gamma$. Since X_1 is efficient and $dX_1 = -17 > w_{\text{opt}} = -\infty$, then we calculate the value of β_1; $\beta_1 = 0 - [(-2)(1) + (-3)(0)] = 2 > 0$. We start exploring the edge E_3; we calculate $\theta^0_3 = \lfloor \text{Min}\{\frac{4}{1}\} \rfloor = 4$.

For $\theta = 4$ (the best value of θ yielding an increase in w), the corresponding solution on the edge E_3 is

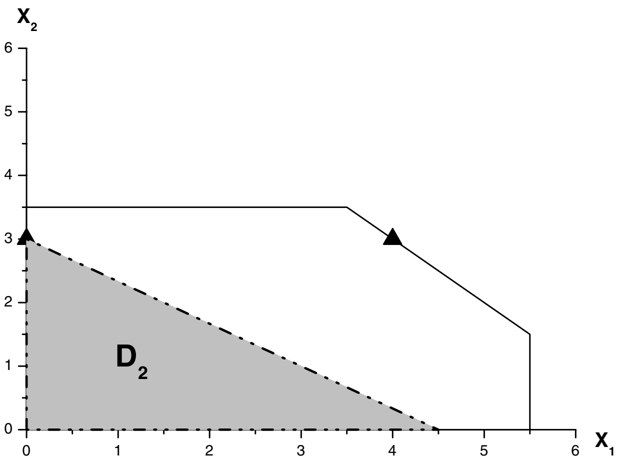

Fig. 2. The reduced region $S_2 = D_2 \cap \mathbb{Z}^2$.

$$\begin{cases} x_1^1 = 4 - 4(1) = 0, \\ x_4^1 = 3 - 4(-2) = 11, \\ x_2^1 = 3 - 4(0) = 3, \\ x_5^1 = 1 - 4(0) = 1, \\ x_3^1 = 4, \\ x_6^1 = 0. \end{cases}$$

The solution $X_1' = (0,3)'$ is being tested for efficiency and we obtain $\psi_1 = \psi_2 = \psi_3 = 0$; $V^* = 0$. Thus X_1' is efficient.

We calculate $w_1' = (-2,-3)\begin{pmatrix} 0 \\ 3 \end{pmatrix} = -9$. As $w_1' > w_{\text{opt}} = -17$, then $w_{\text{opt}} = -9$ and $X_{\text{opt}} = (0,3)'$.

Let $\mathbf{k} := \mathbf{k} + 1 = \mathbf{2}$, we cut by $-2x_1 - 3x_2 \geqslant -9$ (see Fig. 2).

After adjusting the tableau above for the reduced feasible region, and applying the dual simplex method and Gomory method, the optimal feasible solution is $X_2 = (0,3)'$, which is efficient. It corresponds to $w^2 = -9$; $w_{\text{opt}} = -9$ and $X_{\text{opt}} = (0,3)'$ (see Tableau II).

$I_2 = \{1,2,3,4,5\}$, $N_2 = \{6,7\}$.

$\Gamma_2 = \{j \in N_2 | z_{1,j}^2 - c_j^1 \geqslant 0 \text{ and } w_j^2 - d_j^2 \leqslant 0\} = \{6,7\} \neq \emptyset$. Let $\gamma = \Gamma_2$.

Tableau II

Basis	Value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	0	1	0	0	0	0	$-\frac{3}{2}$	$\frac{1}{2}$
x_4	11	0	0	0	1	0	3	-1
x_2	3	0	1	0	0	0	1	0
x_5	1	0	0	0	0	1	-2	0
x_3	4	0	0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
$Z_{1,j}^2 - c_j^1$	6	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$w_j^2 - d_j^2$	-9	0	0	0	0	0	0	-1

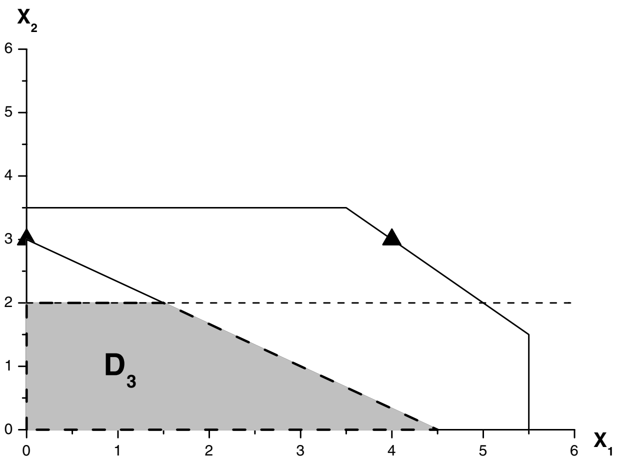

Fig. 3. The reduced region $S_3 = D_3 \cap \mathbb{Z}^2$.

Tableau III

Basis	Value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1	1	1	0	0	0	0	0	0	−2	1
x_4	9	0	0	0	1	0	0	0	4	−2
x_2	2	0	1	0	0	0	0	0	1	0
x_5	3	0	0	0	0	1	0	0	−2	0
x_3	4	0	0	1	0	0	0	0	1	−1
x_6	1	0	0	0	0	0	1	0	−1	0
x_7	1	0	0	0	0	0	0	1	1	−2
$Z_j^3 - c_j^1$	5	0	0	0	0	0	0	0	0	1
$w_j^3 - d_j^3$	−8	0	0	0	0	0	0	0	1	−2

Let $j_2 = 6; \theta_6^0 = 3$. Since X_2 is efficient and $dX_2 = w_{\text{opt}} = -9$, then we calculate the value of β_2; $\beta_2 = 0 - [(-2)(-\frac{3}{2}) + (-3)(1)] = 0$. We do not explore the edge E_{j_2}.

Let $\gamma := \gamma \backslash \{6\}$ and consider the second index $j_2 = 7 \in \gamma$; $\theta_7^0 = \left\lfloor \text{Min}\left\{\frac{0}{\frac{1}{2}}\right\}\right\rfloor = 0$. No integer efficient solution exists in this direction.

$\gamma := \gamma \backslash \{7\} = \emptyset$, then there is no incident edge containing efficient solution.

Let $\mathbf{k} = \mathbf{3}$ and we cut the current feasible region by $\sum_{j \in N_2 \backslash \{7\}} x_j \geqslant 1 \Longleftrightarrow x_6 \geqslant 1$.

From the third line of the simplex matrix in Tableau II, we can write the equation $x_2 + x_6 = 3 \Rightarrow x_6 = 3 - x_2 \Longleftrightarrow 3 - x_2 \geqslant 1 \Longleftrightarrow x_2 \leqslant 2$ (see Fig. 3).

We add this constraint at the bottom of Tableau II and apply the dual simplex method to obtain Tableau III. The solution found is $X_3 = (1, 2)'$ and after solving the problem $(P(X_3))$ for testing efficiency, we find that $\psi_1 = 4$, $\psi_2 = 4$, $\psi_3 = 0$, $x_1 = 3$, $x_2 = 3$ and $V^* = 8$, then X_3 is not efficient.

Now, $I_3 = \{1, 2, 3, 4, 5, 6, 7\}$, $N_3 = \{8, 9\}$.

$\Gamma_3 = \{9\} \neq \emptyset$. Let $\gamma = \Gamma_3$.

Let $j_3 = 9 \in \gamma$ and calculate $\theta_9^0 = \lfloor \text{Min}\{\frac{1}{1}\} \rfloor = 1$.

As X_3 is not efficient, we do not calculate β.

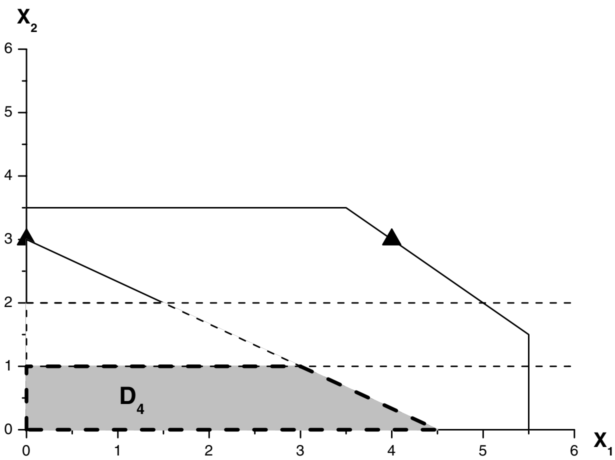

Fig. 4. The feasible region $S = D_4 \cap \mathbb{Z}^2$.

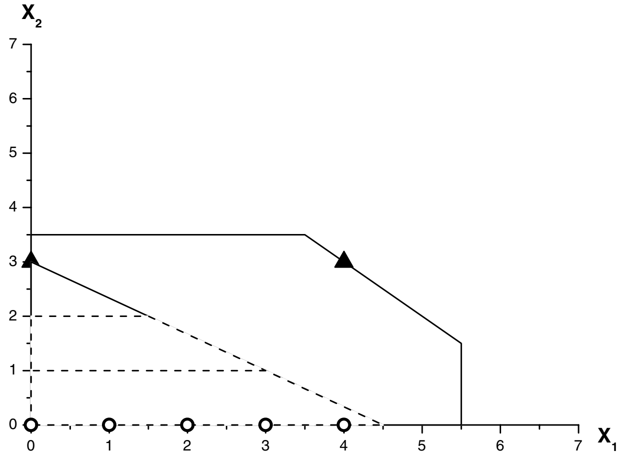

Fig. 5. The feasible domain S_5 represented by the circles.

Tableau IV

Basis	Value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	3	1	0	0	0	0	0	0	0	1	−2
x_4	5	0	0	0	1	0	0	0	0	0	4
x_2	1	0	1	0	0	0	0	0	0	0	1
x_5	5	0	0	0	0	1	0	0	0	0	−2
x_3	3	0	0	1	0	0	0	0	0	−1	1
x_6	2	0	0	0	0	0	1	0	0	0	−1
x_7	0	0	0	0	0	0	0	1	0	−2	1
x_8	1	0	0	0	0	0	0	0	1	0	−1
$Z^4_{1,j} - c^1_j$	0	0	0	0	0	0	0	0	0	1	0
$w^4_j - d^4_j$	−9	0	0	0	0	0	0	0	0	−2	1

Tableau V

Basis	Value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
x_1	4	1	0	0	0	0	0	0	0	0	0	−3	1
x_4	3	0	0	0	1	0	0	0	0	0	0	4	−2
x_2	0	0	1	0	0	0	0	0	0	0	0	1	0
x_5	7	0	0	0	0	1	0	0	0	0	0	−2	0
x_3	3	0	0	1	0	0	0	0	0	0	0	1	−1
x_6	4	0	0	0	0	0	1	0	0	0	0	−1	−1
x_9	1	0	0	0	0	0	0	0	0	1	0	0	−1
x_8	2	0	0	0	0	0	0	0	1	0	0	−1	0
x_{10}	1	0	0	0	0	0	0	0	0	0	1	−1	0
x_7	1	0	0	0	0	0	0	1	0	0	0	1	−2
$Z^5_{1,j} - c^1_j$	4	0	0	0	0	0	0	0	0	0	0	0	1
$w^5_j - d^5_j$	−8	0	0	0	0	0	0	0	0	0	0	1	−2

For $\theta = 1$, the corresponding feasible solution on the edge E_9 is $X'_3 = (0, 2)'$, which is not efficient (since $\psi_1 = 4$, $\psi_2 = 4$, $\psi_3 = 0$, $x_1 = 2$, $x_2 = 3$ and $V^* = 8$). We have $\gamma := \gamma \backslash \{9\} = \emptyset$.

Let $\mathbf{k} = \mathbf{4}$ and cut by $x_8 \geqslant 1$ or, equivalently, $x_2 \leqslant 1$; we obtain the reduced region as shown in Fig. 4.

$X_4 = (3, 1)'$, which is not efficient ($V^* = 8 \neq 0$).

$I_4 = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $N_4 = \{9, 10\}$; $\Gamma_4 = \{9\} \neq \emptyset$.

Let $\gamma = \Gamma_4$ and let $j_4 = 9 \in \gamma$; $\theta_9^0 = \lfloor \mathrm{Min}\{\frac{0}{1}\} \rfloor = 0$. No integer feasible solution in this direction.

$\gamma := \gamma \backslash \{9\} = \emptyset$.

Let $\mathbf{k} = \mathbf{5}$ and cut by $\sum_{j \in N_4 \backslash \{9\}} x_j \geqslant 1 \iff x_{10} \geqslant 1$, which is equivalent to $x_2 \leqslant 0$ (as shown in Fig. 5).

By adding this constraint at the bottom of Tableau IV and solving the new problem, we obtain Tableau V.

$X_5 = (4, 0)'$, ($V^* = 13 \neq 0$) which is not efficient.

$I_5 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $N_5 = \{11, 12\}$. $\Gamma_5 = \{12\} \neq \emptyset$. Let $\gamma = \Gamma_5$ and we take $j_5 = 12 \in \gamma$; we have, $\theta_{12}^0 = \lfloor \mathrm{Min}\{\frac{4}{1}\} \rfloor = 4$;

For $\theta = 4$, $X'_5(4) = (0, 0)'$ ($V^* = 18 \neq 0$) $\Rightarrow X'_5(4)$ is not efficient; for $\theta = 3$, $X'_5(3) = (1, 0)'$ ($V^* = 16 \neq 0$) $\Rightarrow X'_5(3)$ is not efficient; for $\theta = 2$, $X'_5(2) = (2, 0)'$ ($V^* = 13 \neq 0$) $\Rightarrow X'_5(2)$ is not efficient; for $\theta = 1 \Rightarrow X'_5(1)$ is not efficient. None of the solutions on edge E_{12} is efficient. $\gamma := \gamma \backslash \{12\} = \emptyset$.

Let $\mathbf{k} = \mathbf{6}$ and cut by $x_{11} \geqslant 1 \iff x_2 \leqslant -1$ out of the feasible region, and the algorithm, then terminates. The optimal solution is then $w_{\mathrm{opt}} = -9$ and $X_{\mathrm{opt}} = (0, 3)'$.

This example was first presented by Gupta [9] to find the set of integer efficient solutions

$$E(P) = \left\{(4, 3)', (5, 2)', (3, 3)', (4, 2)', (5, 1)', (2, 3)', (5, 0)', (1, 3)', (0, 3)'\right\}.$$

However, our algorithm optimizes the linear function $w = -2x_1 - 3x_2$ without having to determine all these solutions but only $E' = \{(4, 3)', (0, 3)'\}$.

6. Conclusion and comments

The proposed algorithm solves problem (P_E) by using classical linear programming procedures without having to enumerate all the efficient solutions. Of course the algorithm may generate several dominated solutions but it provides a shorter way to the optimal one.

A method that would avoid generating dominated solutions would of course be preferable, if such a method exists. The problem is difficult however; it is a favorite topic for theoretical studies and is still open [11].

Our algorithm has been tested on several instances of different sizes. It has in particular been compared to variants that consider stronger cuts of type II and use them in some cases in which we use cuts of type I. It appears—on several examples—that favoring cuts of type I in the design of the algorithm results in quicker reduction of the domain than with cuts of type II, even in their stronger form. Further experimental (or theoretical) validation of this empirical observation is needed.

Another approach to the problem is to try to work out a solution method in the criteria space instead in the decision variable space. This will be the subject of future investigations.

Acknowledgements

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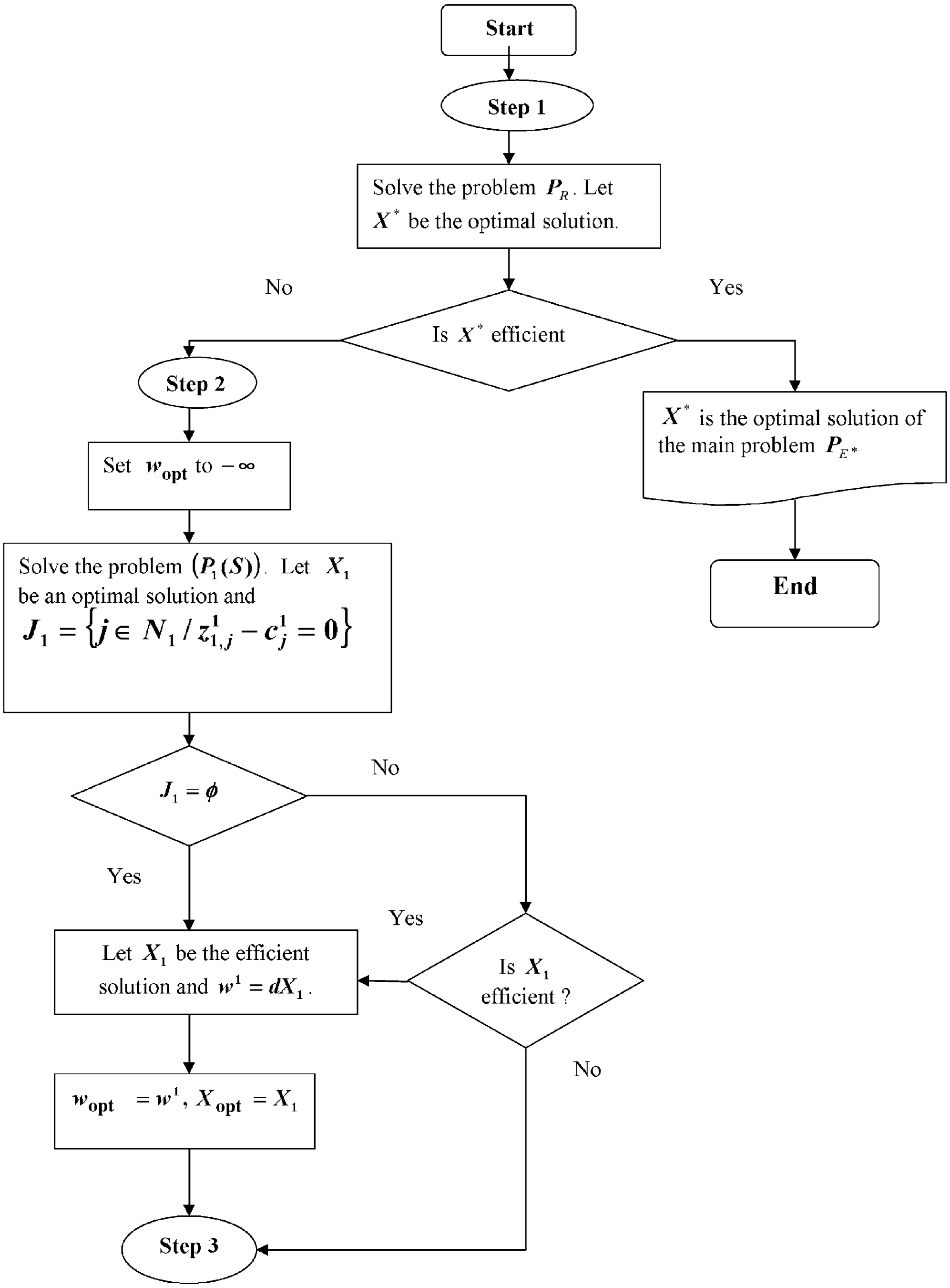
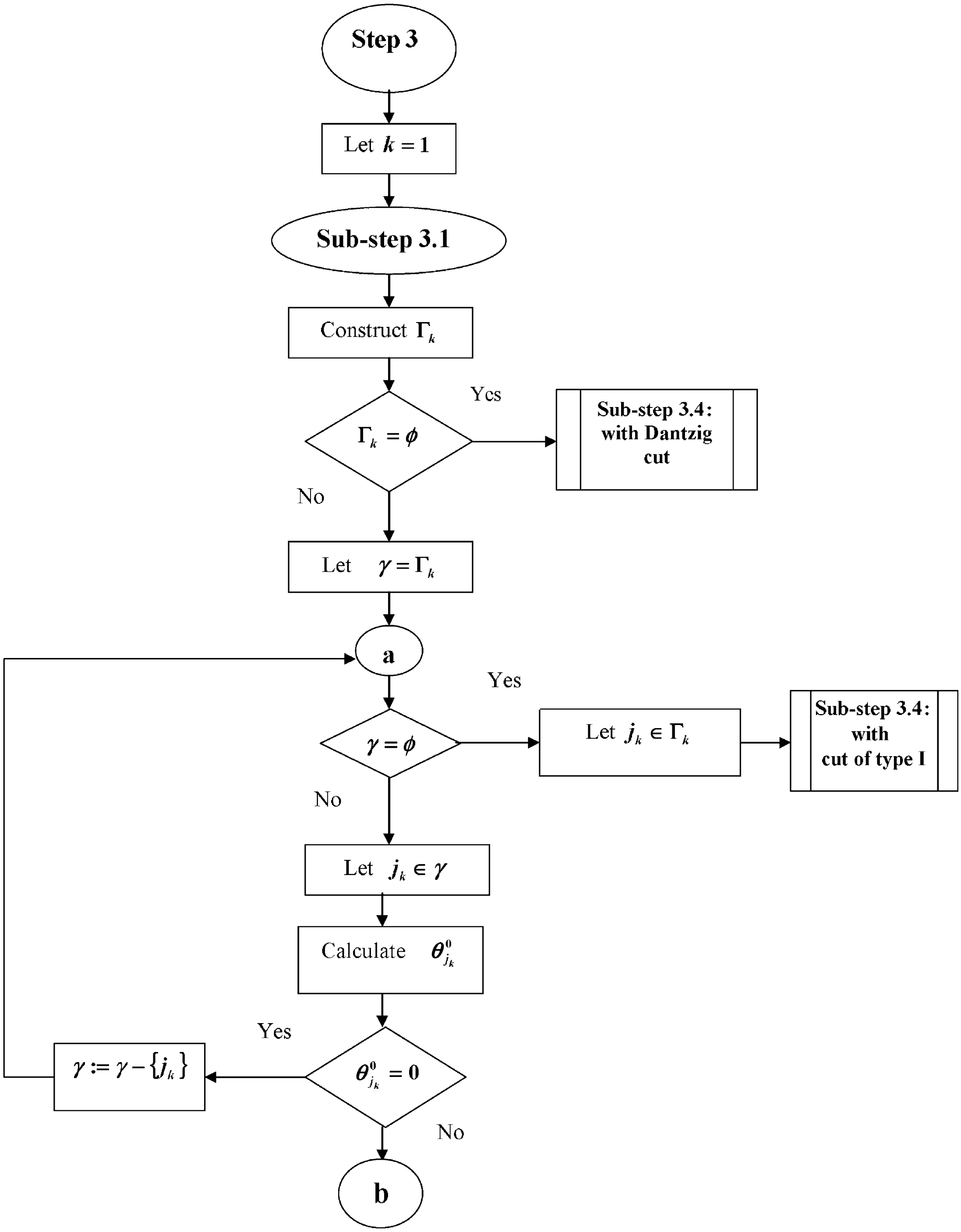
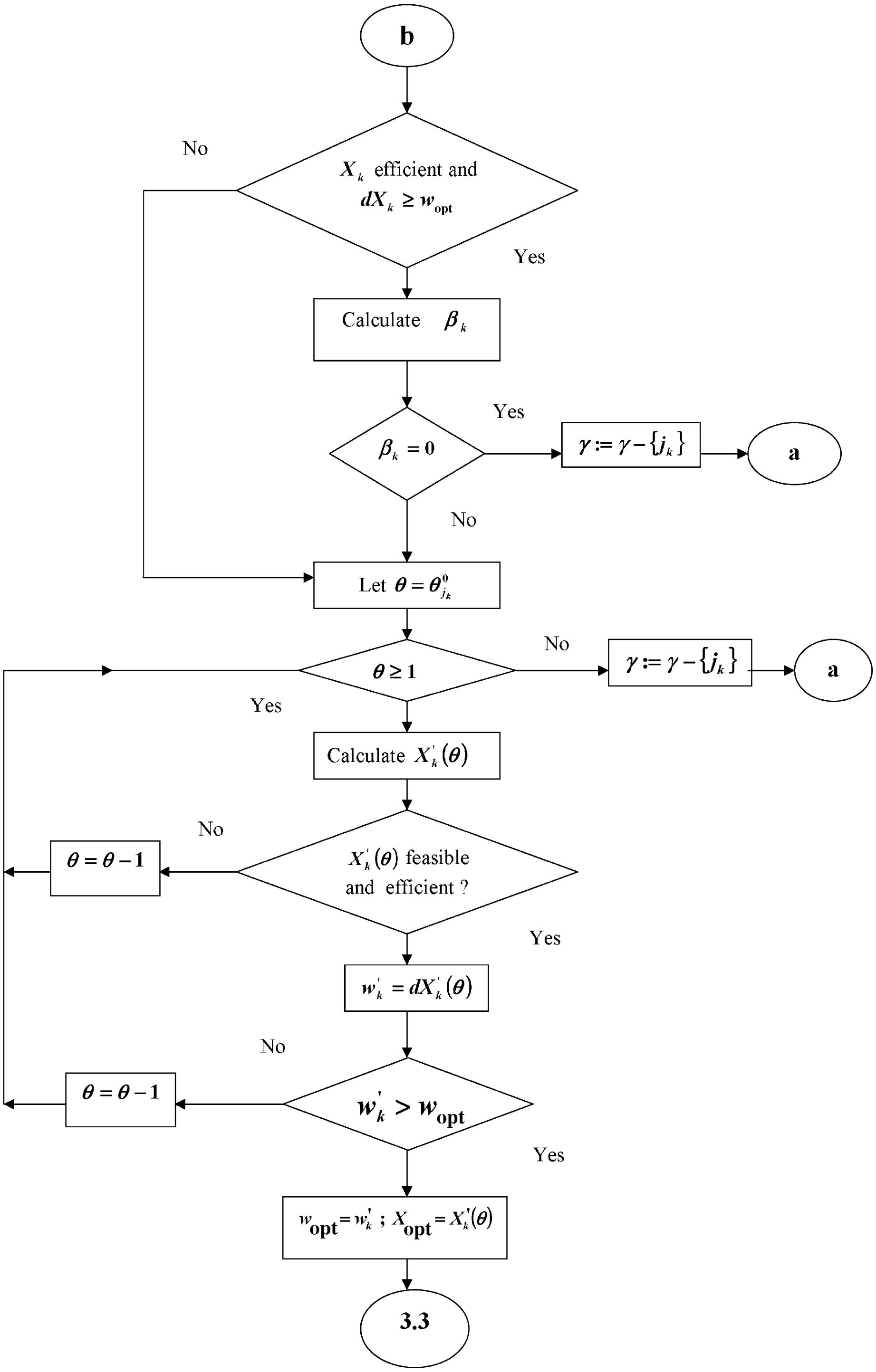
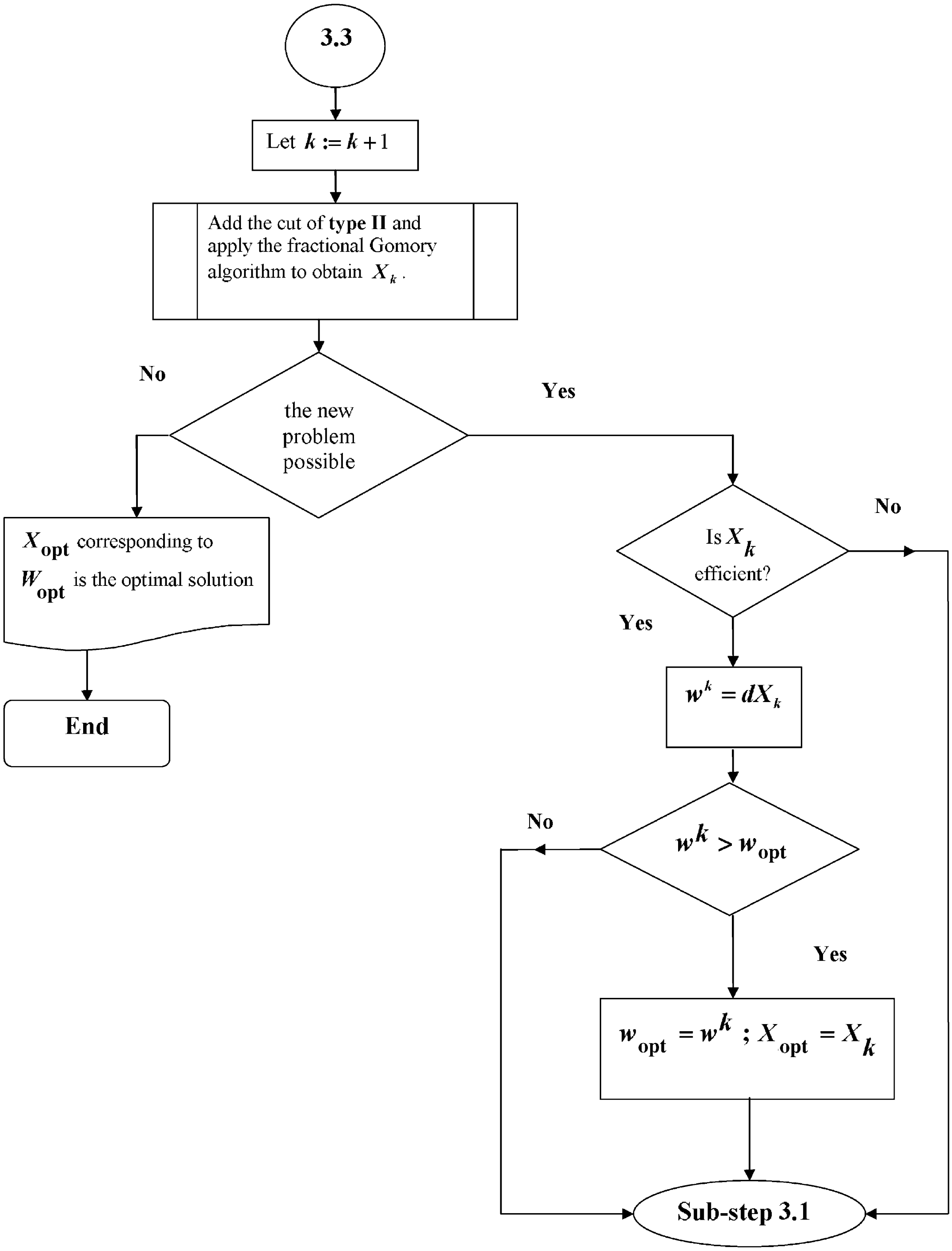
Appendix A

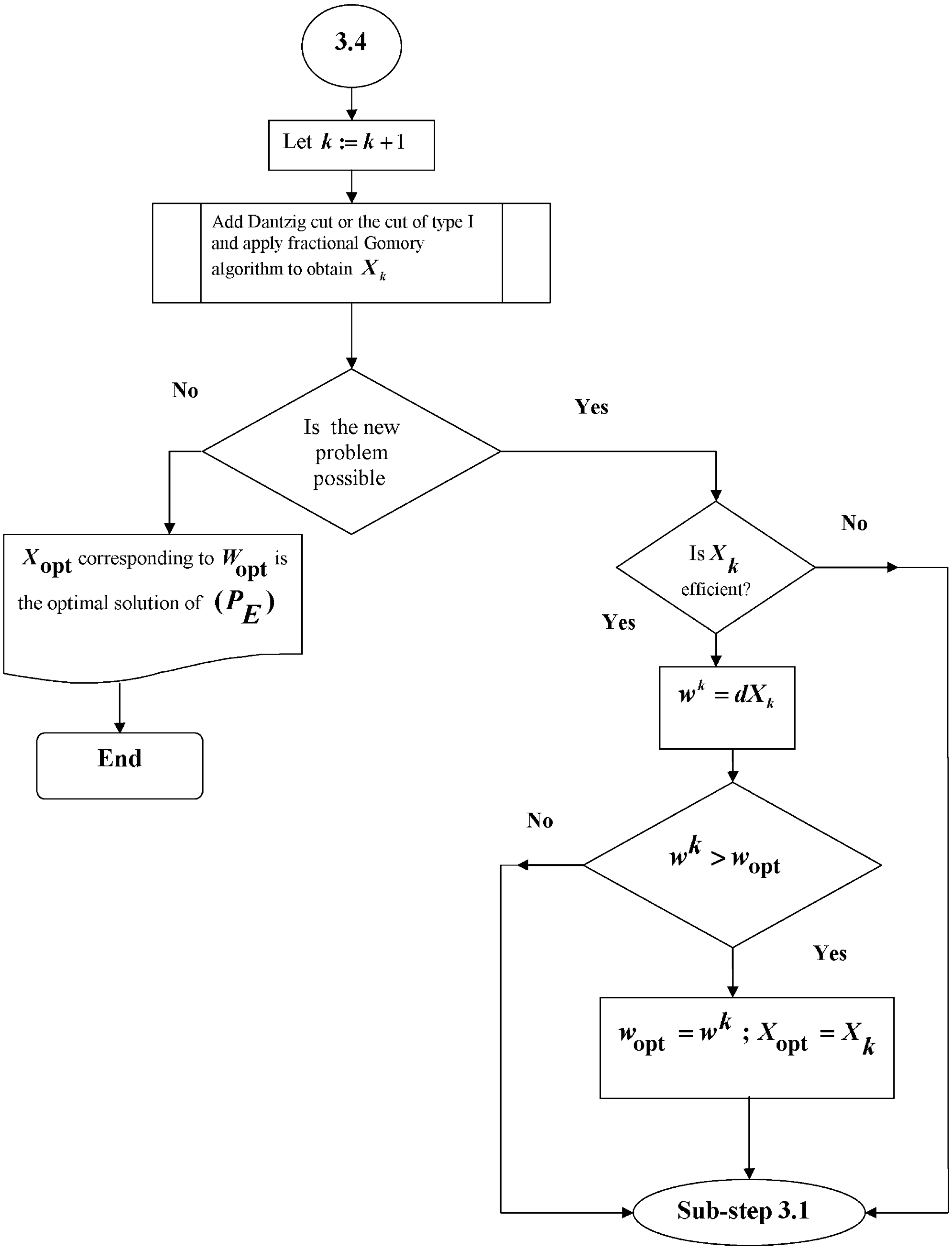


Start
Step 1
Solve the problem P_R. Let X^* be the optimal solution.
No
Yes
Is X^* efficient
Step 2
X^* is the optimal solution of the main problem P_{E^*}
Set w_{opt} to $-\infty$
End
Solve the problem $(P_1(S))$. Let X_1 be an optimal solution and $J_1 = \{j \in N_1 / z^1_{1,j} - c^1_j = 0\}$
$J_1 = \phi$
No
Yes
Yes
Let X_1 be the efficient solution and $w^1 = dX_1$.
Is X_1 efficient ?
No
$w_{\text{opt}} = w^1, X_{\text{opt}} = X_1$
Step 3


Step 3
Let $k = 1$
Sub-step 3.1
Construct Γ_k
$\Gamma_k = \phi$
Ycs
No
Sub-step 3.4: with Dantzig cut
Let $\gamma = \Gamma_k$
a
Yes
$\gamma = \phi$
No
Let $j_k \in \Gamma_k$
Sub-step 3.4: with cut of type I
Let $j_k \in \gamma$
Calculate $\theta^0_{j_k}$
Yes
$\theta^0_{j_k} = 0$
No
$\gamma := \gamma - \{j_k\}$
b


b
X_k efficient and
$dX_k \geq w_{\text{opt}}$
No
Yes
Calculate β_k
$\beta_k = 0$
Yes
No
$\gamma := \gamma - \{j_k\}$
a
Let $\theta = \theta^0_{j_k}$
$\theta \geq 1$
No
Yes
$\gamma := \gamma - \{j_k\}$
a
Calculate $X'_k(\theta)$
$X'_k(\theta)$ feasible
and efficient ?
No
Yes
$\theta = \theta - 1$
$w'_k = dX'_k(\theta)$
$w'_k > w_{\text{opt}}$
No
Yes
$\theta = \theta - 1$
$w_{\text{opt}} = w'_k$; $X_{\text{opt}} = X'_k(\theta)$
3.3


3.3
Let $k := k + 1$
Add the cut of **type II** and apply the fractional Gomory algorithm to obtain X_k.
the new problem possible
No
Yes
X_{opt} corresponding to W_{opt} is the optimal solution
End
Is X_k efficient?
No
Yes
$w^k = dX_k$
$w^k > w_{opt}$
No
Yes
$w_{opt} = w^k$; $X_{opt} = X_k$
Sub-step 3.1


3.4
Let $k := k + 1$
Add Dantzig cut or the cut of type I and apply fractional Gomory algorithm to obtain X_k
Is the new problem possible
No
Yes
$X_{\mathbf{opt}}$ corresponding to $W_{\mathbf{opt}}$ is the optimal solution of (P_E)
End
Is X_k efficient?
No
Yes
$w^k = dX_k$
$w^k > w_{\mathbf{opt}}$
No
Yes
$w_{\mathbf{opt}} = w^k$; $X_{\mathbf{opt}} = X_k$
Sub-step 3.1

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