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The augmented weighted Tchebychev norm for optimizing a linear function over an integer efficient set of a multicriteria linear program

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Abstract

In this paper, we propose a new exact algorithm, using an augmented weighted Tchebychev norm, for optimizing a linear function on the efficient set of a multiple objective integer linear programming problem. This norm is optimized progressively by improving the value of the linear criteria and going through some efficient solutions. The method produced not only the best efficient solution of the linear objective function but also a subset of nondominated solutions that can help decision makers to select the best decision among a large set of Pareto solutions.

Keywords: integer programming; multiple objective; Tchebychev metrics

1. Introduction

In the past two decades, researchers and practitioners have shown increased interest in the problem of optimizing a linear function on the efficient set of multiple objective linear programming problem (MOLP). Several methods and algorithmic ideas have been developed—in general, these approaches can be classified and grouped according to the methodological concepts—which include, among others, adjacent vertex search technique (Philip, 1972; Ecker and Song, 1994; Fülöp, 1994), non-adjacent methods (White, 1996; Dauer and Fosnaugh, 1995), dual approach (Thach et al., 1996), etc. An overview of these approaches can be found in Yamamoto (2002).

In addition to the continuous case, few algorithms have been suggested for solving the problem involving discrete decision variables. For the first time Nguyen (1992) made an attempt to optimize on the integer efficient set, where only an upper bound value for the main objective is proposed. The

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exact algorithm was developed by Abbas and Chaabane (2006) based on a simple selection technique that improves the main objective value at each iteration. Two types of cuts are used and performed successively until the optimal value is obtained and the current truncated region had no integer feasible solution (see also Chaabane, 2007). Jorge (2009) developed another approach that defines a sequence of progressively more constrained single-objective integer problems that successively eliminates undesirable points. The most recent work on this was conducted by Chaabane and Pirlot (2010), which combines ideas from the above-mentioned exact algorithms. In this paper, we propose an exact algorithm, using an augmented weighted Tchebychev norm, to reduce the admissible region by the successive addition of constraints. Subprograms that were integrated in the above methods are now avoided. The technique produced two outputs: an optimal efficient solution and a subset of efficient solutions.

Consider the multiple objective integer linear programming (MOILP) problem

$$(P) \quad V\max\{Cx, \quad x \in D\}, \tag{1}$$

where $D = S \cap \mathbb{Z}$ with $S = \{x \in \mathbb{R}^n \mid Ax \leq b;\ x \geq 0\}$ is a nonempty bounded polyhedron; $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^{m\times 1}$, $C = (c^i)_{i\in\{1,\dots,p\}} \in \mathbb{Z}^{p\times n}$ and $p \geq 2$.

Unlike single-objective problems, the resolution of multiple criteria problems imposes a set of feasible solutions, using the property that no improvement on any criterion is possible without sacrificing on at least one other criterion. These solutions are called efficient solutions or nondominated solutions, which are defined as follows:

A feasible solution $\hat{x} \in D$ is said to be an "efficient solution" of (P) if and only if there is no feasible solution $x \in D$ such that $Cx \geq C\widehat{x}$ and $Cx \neq C\widehat{x}$ ($c^i x \geq c^i \hat{x}$ for all $i = 1, \dots, p$ and $c^i x > c^i \hat{x}$ for at least one i). The point $\hat{z} = C\hat{x}$ is then called "nondominated vector." Otherwise, $\hat{x}$ is not efficient and $\hat{z} = C\hat{x}$ is said to be dominated by $z = Cx$.

$\hat{x} \in D$ is called weakly efficient if there is no $x \in D$ such that $Cx > C\hat{x}$, i.e., $c^i x > c^i \hat{x}$ for all $i = 1, \dots, p$. The point $\hat{z} = C\hat{x}$ is then called weakly nondominated objective vector.

$E(P)$ and $Z(P)$ will be used henceforth to denote, respectively, the set of all efficient solutions of problem (P) and their images in the outcome space defined by the objective vector functions.

A supported efficient solution is an optimal solution of the weighted single-objective problem,

$$\max\{\lambda^t Cx \mid x \in D\},$$

where $\lambda \in R^p_+$ is a weight vector with strictly positive components, λ_i, $i = 1, \dots, p$.

The problem of optimization over the efficient set of the MOILP problem (P) is given by

$$(P_E) \qquad \max\{\phi = d^t x;\ x \in E(P)\}, \tag{2}$$

where $\phi(x) = d^t x$ is linear and called "main objective," $d \in \mathbb{Z}^n$. This problem is very difficult because of nonconvexity of D and absence of information about the admissible region $E(P)$ (the problem is to be solved without solving (P)).

Consider the relaxed problem

$$(P_R) \qquad \max\{\phi = d^t x,\ x \in D\}. \tag{3}$$

Generally, $E(P) \neq D$. Otherwise, if (D) is completely efficient, $E(P)$ can be substituted by D and, in such cases, solving (P_E) is equivalent to solving (P_R).

When $Z(P)$ is not uniformly dominant (the image of D in the criteria space has at least one weakly nondominated solution), a weighted Tchebychev program is not appropriate for our approach; however, an augment weighted Tchebychev metrics can be used (see Steuer, 1986). This process will allow nondominated vector solutions to be characterized and overcome the delicate case of unsupported solutions (see Bowman, 1976). In the next section, we briefly review some basic definitions, results, and foundations of Tchebychev norms. In Section 3, the algorithm is formally presented, relative propositions are provided to prove the finiteness and the convergence properties, and an illustrative example concludes the section. Section 4 describes details of the implementation and computational experiments. Finally, some conclusions and remarks are provided in the last section.

2. Tchebychev norms and preliminary results

The Tchebychev theory, which first originated from Bowman (1976), and its associated properties have been successfully exploited. A. Wierzbicki was one of the first authors to use a Tchebychev-based achievement scalarizing function for reference-point multiobjective programming methods (see Wierzbicki, 1980), within the scope of interactive algorithms for multiple objective optimization (see, for example, Steuer and Choo, 1983, and for the more recent version of Steuer's Tchebychev algorithm, Luque et al., 2010), in algorithms for nonlinear integer bicriteria problems (Eswaran et al., 1989) for solving biobjective integer programs (Ralphs et al., 2006) and other methods (Alves and Clímaco, 2000; Karaivanova et al., 1993; Neumayer and Schweigert, 1994; Schandl et al., 2001).

Moreover, Bowman (1976) proved that the Tchebychev norm's scalarization is appropriate for generating the nondominated objective vectors set, particularly for those that are unsupported. The range of the nondominated objective vectors in the feasible objective region provides valuable information about MOIL problem P provided the objective functions are bounded on the feasible region. Upper bounds of the nondominated solutions set are available in the ideal objective vector $z^\star \in \mathbb{R}^p$. Its components $z_i^\star$ are obtained by maximizing each of the objective functions individually in the feasible region D. A vector strictly better than $z^\star$ is called a utopian objective vector $z^{\star\star}$. We consider Δ the weighting vectors space such that

$$\Delta = \left\{ \beta \in \mathbb{R}^p \mid 0 < \beta_i < 1, \sum_{i=1}^{p} \beta_i = 1 \right\}.$$

A weighted Tchebychev norm in $\mathbb{R}^p$ is the max norm (l_∞ norm) defined as

$$\|(z_1, \ldots, z_p)\|_\infty^\beta = \max_{i=1,\ldots,p} \left\{ \beta_i \mid z_i \mid \right\}.$$

The related distance is

$$\|z^{\star\star} - z\|_\infty^\beta = \max_{i=1,\ldots,p} \left\{ \beta_i \mid z_i^{\star\star} - z_i \mid \right\}. \tag{4}$$

Let $z \in Z$ and $\beta \in \Delta$ the associated weight vector, the weighted Tchebychev norm of z consist for measuring the distance between z and the utopian objective vector $z^{\star\star}$. Methods based on this technique select feasible objective vectors with minimum weighted distance from $z^{\star\star}$, i.e., solving the following problem:

$$\min_{z \in Z} \left\{ ||z^{\star\star} - z||_{\infty}^{\beta} \right\}. \quad (5)$$

An equivalent problem to 5 called "weighted Tchebychev program" denoted by $P(\beta)$ (proposed by Bowman, 1976) is defined by

$$(P(\beta)) \begin{cases} \min \quad \omega \\ \omega \geq \beta_i (z_i^{\star\star} - z_i(x)), \quad 1 \leq i \leq p \\ x \in D, \end{cases} \quad (6)$$

where

$$\beta_i = \frac{1}{z_i^{\star\star} - \overline{z}_i} \left[\sum_{i=1}^{p} \frac{1}{z_i^{\star\star} - \overline{z}_i} \right]^{-1} \quad \forall 1 \leq i \leq p, \quad (7)$$

with $\overline{z}_i = c^i \overline{x}$, where $\overline{x}$ is a prefixed vector in D.

We use the utopian vector instead of the ideal objective values in order to avoid dividing by zero in equation (7). Because the components of the matrix C are assumed integer, $z^{\star\star} = z^{\star} + 1$. The following results provide some conditions for characterizing a nondominated solution

Theorem 1 (Steuer, 1986). *Let Z be finite and*

$$M = \{ z \in Z \mid (x, z, \omega) \text{ is a minimal solution of } (P(\beta)) \text{ for some } \beta \in \Delta \}.$$

Then there exists a $\bar{z} \in M$ such that $\bar{z} \in Z(P)$.

Theorem 2 (Bowman, 1976). *$z = C\hat{x}$, $\hat{x} \in D$ is nondominated solution for (P) only if it is a solution to $(P(\beta))$ for some β.*

A visual representation of $(P(\beta))$ for a particular value of β is shown in Fig. 1. The rectangle $(ABCD)$ is called isoquant and z is said to be vertex of the isoquant. ABCD represents the optimal level lines of the Tchebychev norm. z^1 and z^2 lie on the optimal level lines of the Tchebychev norm. The solutions z^1,z^2, and z^3 are optimal for $P(\beta)$ (for some values of β), but z^1 is an unsupported weakly nondominated solution. z^2 and z^3 are supported nondominated solutions.

Theorem 3 (Bowman, 1976). If *$Z(P)$ is uniformly nondominated, then any solution of $(P(\beta))$ is nondominated.*

In many practical problems, weakly nondominated solutions are not desirable, the above characterization is not appropriate. In order to overcome this situation, we use the augmented weighted Tchebychev norm given by

$$|| z^{\star\star} - z ||_{\infty}^{\beta,\rho} = \max_{i=1,\ldots,p} \left\{ \beta_i \mid z_i^{\star\star} - z_i \mid \right\} + \rho \sum_{i=1}^{p} (z_i^{\star\star} - z_i), \quad (8)$$

where ρ is a sufficiently small positive scalar.

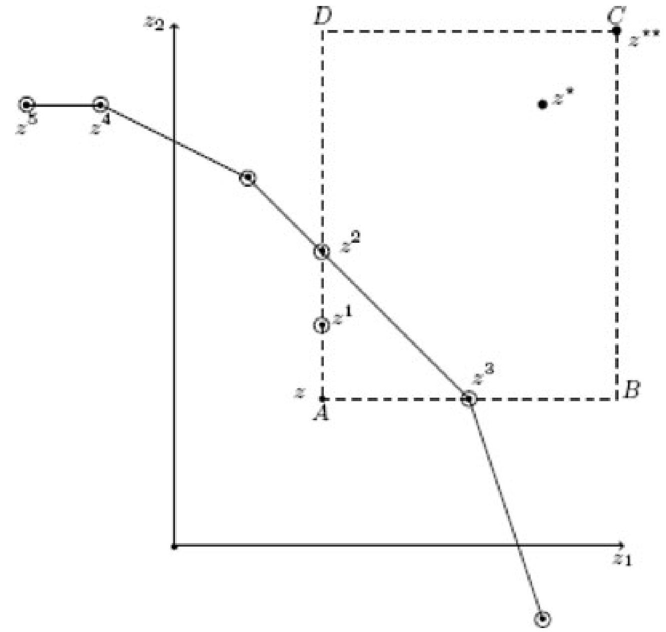

Fig. 1. Weak domination and weighted Tchebychev program.

The linear program under this norm is defined as

$$\left(P_\rho(\beta)\right)\begin{cases}\text{Min} \quad \omega + \rho \sum_{i=1}^{p} (z^{\star\star} - z_i) \\ \omega \geq \beta_i (z_i^{\star\star} - z_i(x)), \quad 1 \leq i \leq p; \\ x \in D.\end{cases} \tag{9}$$

Theoretically, the term $\rho \sum_{i=1}^{p} (z^{\star\star} - \bar{z}_i)$ corresponds to the bound for desirable or acceptable trade-offs. This concept reflects the ratio of change in the values of the objective functions concerning the increment of one objective function that occurs when the value of some other objective function decreases. In this sense, moving from one nondominated vector to another results trade-off, this move indicates a "slight inclination" from some objective function values. The exact value of ρ that works properly depends on the relative size of the optimal objective function values and cannot be computed a priori. The principal aim of introducing the augmented weighted Tchebychev program is to avoid a weakly nondominated solution.

Steuer (1986) has shown that if the ρ is small enough, the augmented weighted Tchebychev program not only guarantees to return a nondominated objective vector but generates any particular nondominated objective vector for an appropriate $\beta \in \Delta$. In practice, precisely in the discrete case, too small values of ρ can cause numeric difficulties, and therefore the values between 0.001 and 0.01 should normally suffice to avoid the problems related to the weakly nondominated solutions and the weighted norm.

Proposition 1. *Let $\beta \in \Delta$, for a small enough fixed $\rho > 0$, any optimal solution to $\left(P_\rho(\beta)\right)$ is a nondominated objective vector to problem (P).*

Proof. Let $\hat{z}$ be an optimal solution to $\left(P_\rho(\beta)\right)$ and suppose that $\hat{z}$ is dominated, then there exists an objective vector $z \in Z$ such that

$$z \geq \hat{z},\ (z_i \geq \hat{z}_i\, \forall i \in \{1, \ldots, p\} \text{ and } \exists k \in \{1, \ldots, p\} \text{ such that } z_k > \hat{z}_k).$$

For $\rho > 0$, we have $\rho \sum_i^p \left(z_i^{**} - z_i\right) < \rho \sum_i^p \left(z_i^{**} - \hat{z}_i\right)$ and

$$\max_{i\in\{1,\ldots,p\}} \left(z_i^{**} - z_i\right) \leq \max_{i\in\{1,\ldots,p\}} \left(z_i^{**} - \hat{z}_i\right).$$

Thus,

$$\|z^{**} - z\|_\infty + \rho \sum_i^p \left(z_i^{**} - z_i\right) < \|z^{**} - \hat{z}\|_\infty + \rho \sum_i^p \left(z_i^{**} - \hat{z}_i\right),$$

which is contradictory to $\hat{z}$ optimal of problem (9). □

Proposition 2. *Let $\hat{z} \in Z$ and $\hat{\beta} \in \Delta$. If $\hat{z}$ is nondominated, then $\hat{z}$* is a unique optimal solution of $P_\rho(\hat{\beta})$ for a small enough fixed $\rho > 0$.

Proof. Let $\hat{z}$ be a nondominated solution to problem (P) and suppose the existence of another optimal solution $\overline{z}$ to problem $P_\rho(\hat{\beta})$. Thus,

$$\overline{w} + \rho \sum_{i=1}^p \left(z_i^{\star\star} - \overline{z}_i\right) < \hat{w} + \rho \sum_{i=1}^p \left(z_i^{\star\star} - \hat{z}_i\right).$$

We also have $\overline{w} > \hat{w}$ (see Steuer, 1986), then

$$\overline{w} + \rho \sum_{i=1}^p \left(z_i^{\star\star} - \overline{z}_i\right) < \hat{w} + \rho \sum_{i=1}^p \left(z_i^{\star\star} - \hat{z}_i\right)$$

$$-\overline{w} < -\hat{w},$$

this implies $\sum_{i=1}^p \left(\overline{z}_i - \hat{z}_i\right) > 0 \Longrightarrow \exists k \in \{1, \ldots, p\} \mid \overline{z}_k > \hat{z}_k$, which contradicts that $(\hat{z})$ is nondominated.

□

3. Description of the method

The proposed algorithm provides a global optimal solution of (P_E) without specifying all efficient solutions of (P). Our technique is articulated on two ideas; we characterize a nondominated objective vector by solving the augmented weighted Tchebychev program $P_\rho(\beta)$ for a sufficiently small value ρ and we reduce progressively the admissible domain by adding more constraints.

Initially, the procedure determines the utopian objective vector $z^{\star\star}$; in the subsequent steps the relaxed problem (P_R) is being solved, an optimal solution x is obtained on which the upper bound of the main criterion is updated and its objective vector $z = Cx$ is referred to as the vertex of the isoquant $z < z^\star$. For small enough value of $\rho > 0$, the augmented weighted Tchebychev program $P_\rho(\beta)$ is solved in order to find the nondominated vector $\hat{z}$ that is closest to the utopian objective vector $z^{\star\star}$, in the direction determined by $z^{\star\star}$ and z. Given that, in the decision space, many efficient solutions may have the same outcome $\hat{z}$, another program is required to find an equivalent efficient solution improving the main objective. A new efficient solution is then generated and added to the current list where the lower bound of ϕ is evaluated. New constraints on the feasible set D of the relaxed problem (P_R) are imposed, without considering whether efficient solutions already generated or any other feasible solutions using dominated objectives vectors. The algorithm terminates when the current feasible space becomes empty or the lower bound coincides with the upper bound.

Assuming that all coefficients of matrix C are integers, at iteration k, the feasible set D is reduced gradually by eliminating all dominated solutions by $C\hat{x}^k$ (see Sylva and Crema, 2004, 2007). The resolution of the following problem enables us to perform this elimination:

$$\left(P_R^k\right) \equiv \max\left\{dx,\, x \in D - \bigcup_{s=1}^{k} D_s\right\}$$

$\{x^s; s = 1, \ldots, k-1\}$ are solutions of (P) obtained at iterations $1, 2, \ldots, k-1$ respectively. Where $D_s = \{x, x \in \mathbb{Z}_+^n, Cx \leq Cx^s\}$ and $\{Cx^s\}_{s=1}^k$ is a subset of nondominated criteria vectors for problem (P).

$$D - \bigcup_{s=1}^{k} D_s$$

$$= \left\{\begin{array}{ll} c^i x \geq (c^i x^s + 1)y_i^s + M_i(1 - y_i^s), & i = 1, 2, \ldots, p;\ s = 1, 2, \ldots, k; \\ \sum_{i=1}^{p} y_i^s \geq 1; & s = 1, 2, \ldots, k \\ y_i^s \in \{0, 1\} & i = 1, 2, \ldots, p;\ s = 1, 2, \ldots, k; \\ x \in D & \end{array}\right\} \tag{10}$$

where M_i is a lower bound for any feasible value of the ith objective function. The associate variables y_i^s, $i = 1, \ldots, p$, of $\hat{x}^s$ and additional constraints are added to impose an improvement on at least one objective function. Note that when $y_i^s = 0$, the constraint is not restrictive and when $y_i^s = 1$, a strict improvement is forced in the ith objective function evaluated at $\hat{x}^s$.

Proposition 3 (Sylva and Crema, 2004). *Let $\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^k$ be efficient solutions to problem (P) and $D_s = \{x \mid x \in \mathbb{Z}_+^n, Cx \leq C\hat{x}^s\}$. Let $\hat{x}^*$ be an efficient solution to the multiple objective integer problem $(P_k) \equiv$ "max"$\{Cx, x \in D - \bigcup_{s=1}^{k} D_s\}$. Then* $\hat{x}^*$ is an efficient solution to problem (P).

The proof is detailed in Sylva and Crema (2004). □

3.1. The algorithm

The technical description of the method is given by

Algorithm 1. Optimizing a linear function over the integer efficient set

Input
$\downarrow A_{(m\times n)}$: matrix of constraints
$\downarrow b_{(m\times 1)}$: RHS vector
$\downarrow d_{(1\times n)}$: main criterion vector
$\downarrow C_{(p\times n)}$: matrix of criteria
Output
$\uparrow x_{\text{opt}}$: optimal solution of the problem (P_E)
$\uparrow \phi_{\text{opt}}$: optimal value of the main criterion ϕ
Initialization
for i=1 to p **do**
  solve $z_i^\star = \max\{c^i x, x \in D\}$; where $z_i^{\star\star} = z_i^\star + 1$ and set the lower bounds $M_i := \min\{c^i x, x \in D\}$
end for
$\phi_{\text{sup}} := +\infty$ and $\phi_{\text{inf}} := -\infty$: upper and lower bounds of ϕ function; $k = 1$,
$E_1 := \emptyset$; $\overline{D} := D$; $end := false$
while $end = false$ **do**
  Solve $(P_R^k) \equiv \max\{dx, x \in \overline{D}\}$
  if (P_R^k) is unfeasible or $\phi_{\text{inf}} \geq \phi_{\text{sup}}$ **then**
    x_{opt} an optimal solution of (P_E); $end := true$; **Terminate**
  else
    Let x^k be an optimal solution of (P_R^k)
    Let $\phi_{\text{sup}} = dx^k$, compute the weighted vector β^k of $z^k = Cx^k$
    Let $(\hat{x}^k, \hat{z}^k)$ be an optimal solution of $P_\rho(\beta^k)$
    if $d\hat{x}^k = \phi_{\text{sup}}$ **then**
      $x_{\text{opt}} = \hat{x}^k$; $\phi_{\text{opt}} = \phi_{\text{sup}}$; $end := true$; **Terminate**
    else
      Solve $Q(\hat{z}^k) \equiv \max\{dx \mid x \in D, Cx = \hat{z}^k\}$
      Let $\bar{x}^k$ be an optimal solution of $Q(\hat{z}^k)$
      if $d\bar{x}^k > \phi_{\text{inf}}$ **then**
        $x_{\text{opt}} := \bar{x}^k$, $\phi_{\text{inf}} := d\bar{x}^k$ and $\phi_{\text{opt}} := \phi_{\text{inf}}$; let $E_{k+1} = E_k \cup \{\bar{x}^k\}$,
        $k = k + 1$ and $\overline{D} := D \setminus \cup_{s=1}^{k-1} D_s$; $D_s = \{x \in \mathbb{Z}^n \mid Cx \leq C\bar{x}^s; \bar{x}^s \in E_{k-1}\}$
      else
        if $\phi_{\text{inf}} \geq \phi_{\text{sup}}$ **then**
          x_{opt} is an optimal solution of (P_E) and ϕ_{opt} the optimal value of ϕ; $end := true$: **Terminate**
        else
          $x_{\text{opt}} := \bar{x}^k$, $\phi_{\text{inf}} := d\bar{x}^k$ and $\phi_{\text{opt}} := \phi_{\text{inf}}$; let $E_{k+1} = E_k \cup \{\bar{x}^k\}$,
          $k = k + 1$ and $\overline{D} := D \setminus \bigcup_{s=1}^{k-1} D_s$; $D_s = \left\{x \in \mathbb{Z}^n \mid Cx \leq C\bar{x}^s; \bar{x}^s \in E_{k-1}\right\}$
        end if
      end if
    end if
  end if
end while

Proposition 4. *The algorithm above converges in a finite number of iterations.*

Proof. Since D is a finite bounded set, the number of efficient solutions $|E(P)|$ is finite. At each iteration of the algorithm, a new improved efficient solution is generated and the admissible region is being reduced there until infeasibility. Thus, the procedure converges to the optimal solution of (P_E) in a finite number of iterations. □

3.2. A didactic example

Consider the following MOILP problem:

$$(P)\left\{\begin{array}{l} \text{“max”}\quad z_1 = x_1 + 3x_2 \\ \text{“max”}\quad z_2 = -3x_1 - x_2 \\ D\left\{\begin{array}{l} -2x_1 + 5x_2 \leq 23 \\ 4x_1 + x_2 \leq 31 \\ x_1 - x_2 \leq 4 \\ -x_1 - 3x_2 \leq -8 \\ -3x_1 - x_2 \leq -8 \\ x_1,\quad x_2 \in \mathbb{Z}_+^* \end{array}\right. \end{array}\right. \tag{11}$$

and the main problem

$$(P_E) \equiv \max\{\phi = x_1 - 4x_2 \mid (x_1 \quad x_2)' \in E(P)\}.$$

The relaxed problem (P_R) is given by

$$(P_R) \equiv \max\{\phi = x_1 - 4x_2 \mid (x_1 \quad x_2)' \in D\}.$$

We can use one of the algorithms developed in Sylva and Crema (2004) to find the efficient set $E(P)$. This example contains six unsupported efficient solutions out of eight efficient solutions (see Fig. 2). For this example, the parameter ρ has been fixed at 0.004.

Step 0 Initialization.

- Ideal point $z^\star = (27 \quad -8)'$; $z^{\star\star} = (28 \quad -7)'$; the lower bounds of the objective functions are $M_1 = 0$, $M_2 = -25$;
- $\phi_{\text{inf}} := -\infty$, $\phi_{\text{sup}} := +\infty$, $k = 1$, $E(P)_1 := \emptyset$ and $end := false$.

First iteration

We solve the relaxed problem $P_R^1 \equiv \max\{\phi \mid x \in D\}$.

An optimal solution is $x^1 = (5 \quad 1)'$. Let $z^1 = Cx^1 = (8 \quad -16)'$ be its image in the outcome space criteria, $\phi_{\text{sup}} = dx^1 = 1$ and $\phi_{\text{inf}} \ngeq \phi_{\text{sup}}$. We compute the weighted vector β^1 of z^1, $\beta^1 = (0.3103, 0.6897)$.

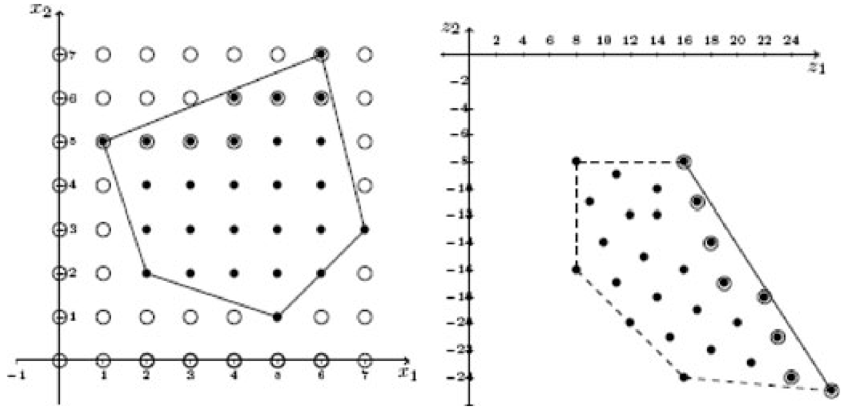

Fig. 2. Decision and outcome space.

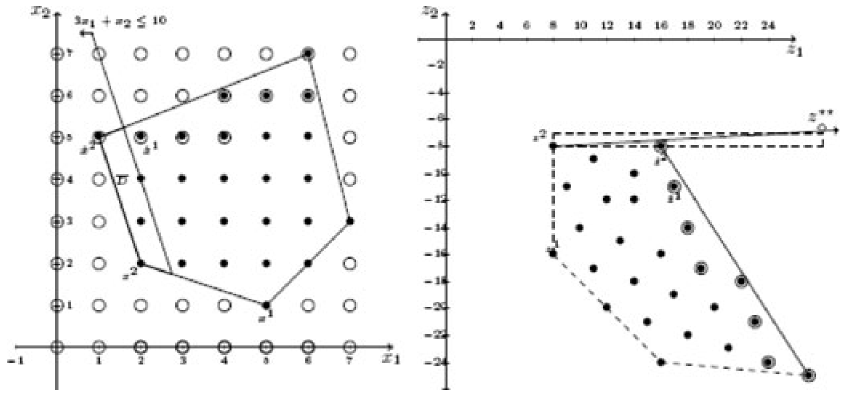

Fig. 3. Second iteration.

We solve the generalized Tchebychev program $P_\rho(\beta^1)$, which is defined as follows:

$$P_\rho(\beta^1)\begin{cases}\min \omega - 0.004(-2x_1 + 2x_2)\\ \quad\omega \geq 0.3103(28 - x_1 - 3x_2)\\ \quad\omega \geq 0.6897(-7 + 3x_1 + x_2)\\ x_1, x_2 \in D\\ \quad\omega \geq 0.\end{cases} \tag{12}$$

The solution $\hat{z}^1 = (17 \quad -11)'$ is a nondominated point with minimal weighted Tchebychev distance, we obtain $\hat{x}^1 = (2 \quad 5)'$ and $\phi(\hat{x}^1) = -18 \neq \phi_{\text{sup}}$. We solve $Q(\hat{z}^1) \equiv \max\{dx, Cx = \hat{z}^1, x \in D\}$. $\bar{x}^1 = (2 \quad 5)' = \hat{x}^1$; $\phi_{\text{inf}} = -18 \not\geq \phi_{\text{sup}} = 1$; and $E(P)_2 = \{\bar{x}^1\}$.

The second iteration is shown in Fig. 3 and the remaining iterations (for $k \geq 2$) are summarized in Table 1 and represented in Fig. 4. $(x_{\text{opt}}, \phi_{\text{opt}}) = ((4 \quad 5)', -16)$ is the optimal solution that is obtained in the fourth iteration.

Table 1
The obtained results at each iteration

Iteration k	(P_R) x^k z^k	Value of β_i^k	$\left(P_\rho\left(\beta^k\right)\right)$ $\widehat{x}^k$ $\widehat{z}^k$	$(Q(\hat{z}^k))$ $\overline{x}^k$ $\overline{z}^k$	$\phi_{\text{inf}} \geq \phi_{\text{sup}}$ or (P_R) is unfeasible	OPT. SOL. x_{opt} ϕ_{opt}
1	$(5 \quad 1)'$	0.3103	$(2 \quad 5)'$	$(2 \quad 5)'$	False	$(2 \quad 5)'$
	$(8 \quad -16)'$	0.6897	$(17 \quad -11)'$	$(17 \quad -11)'$		-18
2	$(2 \quad 2)'$	0.0476	$(1 \quad 5)'$	$(1 \quad 5)'$	False	$(2 \quad 5)'$
	$(8 \quad -8)'$	0.9524	$(16 \quad -8)'$	$(16 \quad -8)'$		-18
3	$(6 \quad 4)'$	0.6000	$(4 \quad 6)'$	$(4 \quad 6)'$	False	$(2 \quad 5)'$
	$(18 \quad -22)'$	0.4000	$(22 \quad -18)'$	$(22 \quad -18)'$		-18
4	$(4 \quad 5)'$	0.5263	$(4 \quad 5)'$	$(4 \quad 5)'$	True	$(4 \quad 5)'$
	$(19 \quad -17)'$	0.4737	$(19 \quad -17)'$	$(19 \quad -17)'$		-16

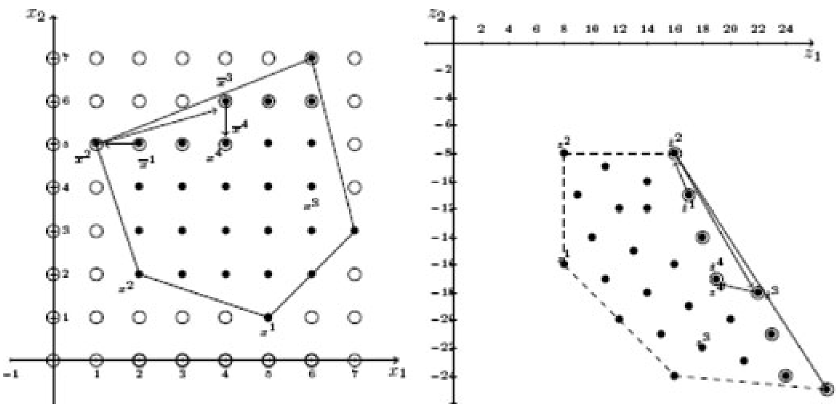

Fig. 4. List of efficient/nondominated solutions.

4. Computational results

The algorithm described above was implemented in the MATLAB environment and run on a PC (Intel Pentium 2.66 GHz processor). We use the CPLEX 12.2 library for solving linear and integer linear programming problems. The main feature of the algorithm lies in the resolution of two specific integer linear programs—the relaxed problem and the augmented weighted Tchebychev program. For their resolution, branch and bound technique is used.

The algorithm was tested with random MOILP problems generated from discrete uniform distribution. The components of the matrices A, C, and the vector b were drawn in the ranges [1, 30], [−20, 20], and [50, 150], respectively. The vector d is generated in the same way as C. To avoid infeasibility, all the constraints of each problem are of kind " $\leq$." Furthermore, since all the coefficients of A and b are positive, the boundedness of the feasible region is assured. The number of objective functions p takes the values 3, 5, and 8. A total of 870 problems are grouped according to the number of variables, constraints, and objective functions into 87 categories; in each category, 10 instances are solved.

Table 2
Computational results

p	$p = 3$		$p = 5$		$p = 8$	
$m \times n$	CPU (Sec.)	Iter	CPU (Sec.)	Iter	CPU (Sec.)	Iter
10 × 10	0.38	2	0.45	1	0.59	2
	[0.26, 0.54]	[1, 3]	[0.36, 0.64]	[1, 3]	[0.54, 1.13]	[1, 5]
10 × 15	0.67	2.5	0.49	1	0.70	1
	[0.32, 1.76]	[1, 7]	[0.46, 1.07]	[1, 4]	[0.64, 0.82]	[1, 2]
15 × 15	0.36	1	0.49	1	0.63	1
	[0.28, 0.70]	[1, 4]	[0.42, 0.67]	[1, 3]	[0.05, 14.86]	[1, 5]
20 × 15	0.57	3.5	0.4715	1	0.67	1
	[0.28, 1.07]	[1, 5]	[0.39, 0.79]	[1, 3]	[0.60, 0.97]	[1, 1]
25 × 15	0.42	2	0.50	1	0.65	1
	[0.30, 0.56]	[1, 3]	[0.40, 0.71]	[1, 3]	[0.58, 1.10]	[1, 2]
25 × 25	0.48	2	0.63	1.5	0.94	1
	[0.31, 0.69]	[1, 3]	[0.46, 1.62]	[1, 5]	[0.71, 1.93]	[1, 8]
35 × 25	0.53	2	0.55	1	0.77	1
	[0.37, 1.31]	[2, 5]	[0.47, 1.06]	[1, 4]	[0.61, 4.15]	[1, 10]
35 × 30	0.66	3	0.58	1	0.92	1
	[0.40, 2.47]	[1, 6]	[0.47, 1.27]	[1, 3]	[0.70, 3.02]	[1, 13]
40 × 30	0.56	2	0.82	2	0.85	1
	[0.40, 0.85]	[1, 4]	[0.50, 2.35]	[2, 7]	[0.71, 2.38]	[1, 3]
40 × 40	0.88	3	0.92	2	1.1952	1.5
	[0.39, 2.64]	[1, 6]	[0.54, 1.80]	[1, 4]	[0.81, 2.60]	[1, 4]
50 × 50	0.91	3	0.86	2	1.12	1
	[0.60, 1.54]	[1, 5]	[0.56, 1.68]	[1, 5]	[0.95,2.54]	[1, 3]
50 × 60	1.01	2.5	2.00	3.5	2.04	1
	[0.57, 5.38]	[1,7]	[0.96, 3.52]	[1, 5]	[1.21, 3.61]	[1, 3]
60 × 60	1.07	2.5	2.13	2	1.85	1
	[0.49, 3.47]	[1, 7]	[0.70, 4.02]	[1, 5]	[1.04, 3.45]	[1, 3]
70 × 60	4.43	5.5	5.07	5	3.50	1.5
	[1.26, 11.41]	[1, 8]	[2.33, 8.67]	[2, 7]	[1.97, 8.53]	[1, 5]
70 × 80	2.71	3	2.83	3.5	2.83	1
	[1.26, 11.41]	[1, 8]	[2.33, 8.67]	[2, 7]	[1.97, 8.53]	[1, 5]
80 × 80	5.60	4.5	8.39	4.5	4.65	2
	[4.06, 16.25]	[3, 7]	[1.29, 29.82]	[1, 9]	[3.23, 9.10]	[1, 5]
80 × 90	1.40	1	2.33	1	3.29	1
	[0.93, 10.24]	[1, 5]	[1.79, 19.84]	[1, 7]	[2.79, 14.95]	[1, 5]
80 × 100	1.64	4	2.66	1	6.62	2
	[1.12, 107.30]	[1, 15]	[1.70, 167.83]	[1, 12]	[4.80, 17.61]	[1, 6]

The results reported in Tables 2 and 3—median CPU time (in seconds), required iterations, and the minimum and maximum values of each measure—show that the proposed algorithm for small and medium dimensions works efficiently; in terms of number of iterations (median of the number of iteration < 6) and execution time (median CPU time < 11 seconds). For higher dimensions, the resolution of such problems becomes difficult due to the factors, such as multiple objective and discrete nature.

Table 3
Computational results

p	$p=3$		$p=5$		$p=8$	
$m \times n$	CPU (Sec.)	Iter	CPU (Sec.)	Iter	CPU (Sec.)	Iter
100×100	8.67	4	9.33	3.5	1.45	1
	[4.21, 21.51]	[1, 7]	[1.37, 143.30]	[1, 13]	[1.33, 14.18]	[1, 10]
100×120	2.04	4	2.18	2.5	2.2659	1
	[1.00, 2.74]	[1, 5]	[1.34, 2.83]	[1, 3]	[1.89, 4.29]	[1, 4]
120×120	2.02	2.5	2.0787	2	2.43	1
	[1.00, 3.78]	[1, 6]	[1.45, 4.36]	[1, 4]	[2.12, 40.11]	[1, 32]
130×140	3.2147	2	3.33	2	3.84	1
	[1.41, 6.56]	[1, 5]	[1.99, 12.58]	[1, 5]	[2.71, 22.43]	[1, 11]
140×150	3.80	2	2.87	1	3.52	1
	[1.62,7.37]	[1, 5]	[2.24, 18.22]	[1, 7]	[3.12, 26.93]	[1, 3]
150×160	4.57	3.5	4.42	2	4.22	1
	[2.92, 9.03]	[1, 5]	[2.76, 13.51]	[1, 5]	[3.35, 5.48]	[1, 2]
160×170	5.41	2.5	5.92	2	4.51	1
	[3.27, 11.09]	[2, 6]	[3.31, 10.90]	[1, 5]	[4.20, 7.33]	[1, 2]
170×180	6.72	3	5.97	2	5.20	1
	[2.65, 15.30]	[1, 6]	[3.54, 9.90]	[1, 3]	[4.74, 12.36]	[1, 4]
180×200	5.20	2.5	9.04	3	6.28	1
	[2.97, 10.94]	[1, 5]	[4.28, 33.63]	[1, 7]	[5.83, 49.08]	[1, 14]
200×200	7.77	3	7.24	2	9.35	2
	[5.70, 11.68]	[2, 5]	[4.64, 10.24]	[1, 2]	[6.42, 12.58]	[1, 3]
200×220	9.66	3	8.99	2	10.54	2
	[4.41, 18.34]	[1, 5]	[5.67, 15.83]	[1, 4]	[7.62, 11.72]	[1, 2]

5. Conclusion

In this paper, we presented an exact algorithm for optimizing a linear function on the efficient set of an MOILP. We achieve this objective by combining two ideas: one consists of solving the augmented weighted Tchebychev program in the outcome space criteria to characterize non-dominated objective solutions, a second stage of optimization is required to find an efficient solution that improves the main objective. The second idea is a process of added cuts for eliminating the dominated solutions found previously, therefore reducing progressively the admissible region.

The algorithm was coded using the `MATLAB` environment utilizing the `CPLEX` 12.2 library. It is tested for several problems randomly generated from a discrete uniform distribution and the results obtained are very encouraging. The computational experience shows that the proposed algorithm is very efficient in terms of the number of efficient solutions gone over (does not exceed a median value of 5) with problems of considerable dimensions (200 constraints, 220 variables, and 8 objectives). For future research, we suggest an updated survey for a comparison between several methods and for the study of an expert system and robustness of the developed algorithms.

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